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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES,
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RELATIONS BETWEEN ORDER AND TOPOLOGY IN VECTOR SPACES

By J. D. WESTON (*Newcastle upon Tyne*)

[Received 5 June 1957]

Introduction

A PARTIALLY ordered vector space which is also a topological vector space usually has some structural property by which the order and the topology are directly related. For example, it very commonly happens that the space has arbitrarily small neighbourhoods of the origin which are intervals (an interval being a set which contains x if it contains u and v with $u \leqslant x \leqslant v$); spaces that have this property have been called *A-spaces* (**1**).† Locally convex A-spaces are of particular interest (**1**, **2**, **3**, **4**); those for which the topology is given by a norm are characterized by the fact that there is a positive real number η such that, if $x \geqslant 0$ and $x' \geqslant 0$, then $\|x+x'\| \geqslant \eta\|x\|$ [cf. (**5**)].

In this paper we study a special type of A-space in which the order and the topology are very closely related. Some of the results suggest that an important part of the theory of partially ordered vector spaces may properly be regarded as a branch of the theory of locally convex spaces.

Order units

The usual definition of a topological vector space (over the field of real numbers) will be assumed (**6**). As regards partially ordered vector spaces, a definition slightly weaker than usual is sufficient for our purposes: it will be assumed only that the positive cone (the set of elements x for which $x \geqslant 0$) satisfies the condition

$$\textit{if } x \geqslant 0,\ x' \geqslant 0, \textit{ and } \alpha \geqslant 0, \textit{ then } x+x' \geqslant 0 \textit{ and } \alpha x \geqslant 0.$$

This gives a transitive order relation when '$x \geqslant x'$', or '$x' \leqslant x$', is defined to mean '$x-x' \geqslant 0$'. It is usual to stipulate also that, if $0 \leqslant x \leqslant 0$, then $x = 0$; here it is enough to remark that this further condition is necessarily satisfied by a Hausdorff A-space (for, if $0 \leqslant x \leqslant 0$ then any interval that contains 0 contains x).

† These spaces are discussed in a memoir by I. Namioka (**13**), which has been published since the present paper was submitted. Namioka's terminology differs somewhat from mine; in particular, he uses the term 'interval' in a more restricted sense, and gives the name 'full set' to what I have called an interval: his 'locally full' spaces are precisely the A-spaces.

A partially ordered vector space which is a topological vector space will be called an *A*-space* if it has arbitrarily small neighbourhoods of the origin which are intervals of the type $\{x: -u \leqslant x \leqslant u\}$. For example, consider the Banach space of all bounded real-valued functions on a given set, with the usual norm and the usual ordering: this, or any subspace of it which contains the constants, is an A^*-space. The real numbers themselves form an A^*-space. It is obvious that an A^*-space is locally convex.

An element u of a partially ordered vector space is called an *order unit* (**7**) if, corresponding to any element x of the space, there is a number $\xi \geqslant 0$ such that $-\xi u \leqslant x \leqslant \xi u$; this is equivalent to the existence, for any x, of a positive number ρ such that $-u \leqslant \rho x \leqslant u$. An order unit may not exist, and, if it does exist, it is not unique (except in the trivial case of a 0-dimensional space). The following theorem shows that the existence of an order unit is a property which has a topological interpretation.

THEOREM 1. *A partially ordered vector space X has an order unit if and only if it admits a topology with which it is an A^*-space. If such a topology exists, then it is unique, being the largest (finest) topology with which X is an A-space; it has the property that an interval $\{x: -u \leqslant x \leqslant u\}$ is a neighbourhood of the origin if and only if u is an order unit.*

Proof. Suppose first that X is an A^*-space, having the interval $\{x: -u \leqslant x \leqslant u\}$ as a neighbourhood of the origin. Since any neighbourhood of the origin in a topological vector space 'absorbs' every point of the space, there corresponds to any point x a positive number ρ such that $-u \leqslant \rho x \leqslant u$. Thus u is an order unit.

Now suppose, on the other hand, that X is a partially ordered vector space with an order unit u, and let I_u be the interval $\{x: -u \leqslant x \leqslant u\}$. For any positive number ρ, the set ρI_u is the interval $\{x: -\rho u \leqslant x \leqslant \rho u\}$; this is a symmetric convex set which absorbs every point of X. The class of all sets ρI_u includes the intersection of any two of its members and is therefore a basis of neighbourhoods of the origin determining a topology $\mathscr{T}$ with which X is an A^*-space. If X has another topology with which it is an A-space, let U be a neighbourhood of the origin with respect to this and suppose that U is an interval. Then, for some positive number ρ, both ρu and $-\rho u$ belong to U, and therefore $\rho I_u \subseteq U$; thus U is a neighbourhood of the origin with respect to $\mathscr{T}$. This shows that $\mathscr{T}$ is larger than any other A-space topology for X, and in particular that $\mathscr{T}$ is the only topology with which X is an A^*-space.

It is known (**2**, **3**) that every continuous linear functional on a locally convex A-space can be expressed as the difference of two continuous linear functionals which are both non-negative (in the sense that their values on the positive cone are non-negative). From this it follows that the weak topology of a locally convex A-space is determined by the non-negative continuous linear functionals: the finite intersections of sets of the type $\{x: |\varphi(x)| < \epsilon\}$, where φ is such a functional and $\epsilon > 0$, form a basis of weak neighbourhoods of the origin. Now, if φ is non-negative, the set $\{x: |\varphi(x)| < \epsilon\}$ is an interval; for, if $u \leqslant x \leqslant v$, then $\varphi(u) \leqslant \varphi(x) \leqslant \varphi(v)$. Hence, if we replace the given topology by the weak topology, we still have an A-space. This fact, in conjunction with Theorem 1, shows that the possession of an order unit is not a sufficient condition for an A-space to be an A^*-space.

Boundedness

In a topological vector space which is partially ordered, there are two notions of boundedness. A set B is *topologically bounded* if it is absorbed by every neighbourhood of the origin (corresponding to any neighbourhood U of the origin, there is a positive number ρ such that $\rho B \subseteq U$); it is *bounded in the sense of order* if there exist u and v such that $u \leqslant x \leqslant v$ for all x in B. In an A-space, the second kind of boundedness implies the first: for a neighbourhood of the origin which is an interval absorbs any set of the type $\{x: u \leqslant x \leqslant v\}$ since it absorbs any set consisting of two points u and v. It follows that an A-space is an A^*-space if it has a single neighbourhood of the origin of the type I_u, where $I_u = \{x: -u \leqslant x \leqslant u\}$; for then the sets ρI_u, for $\rho > 0$, that is the intervals $\{x: -\rho u \leqslant x \leqslant \rho u\}$, are arbitrarily small neighbourhoods of the origin. This remark leads to the following characterization of A^*-spaces and their order units.

THEOREM 2. *An A-space is an A^*-space if and only if the interior of the positive cone is not empty. The order units of an A^*-space are precisely the interior points of the positive cone.*

Proof. Suppose first that u is an order unit in an A^*-space and let I_u be the interval $\{x: -u \leqslant x \leqslant u\}$. By Theorem 1, I_u is a neighbourhood of the origin. Hence $u+I_u$, that is $\{x: 0 \leqslant x \leqslant 2u\}$, is a neighbourhood of u consisting entirely of non-negative points; thus u is an interior point of the positive cone.

Now suppose, on the other hand, that u is an interior point of the positive cone C in an A-space X. Then, since $u = 2u-u$, u is also an interior point of the set $2u-C$, that is, of $\{x: x \leqslant 2u\}$. Hence u is an

interior point of the interval $\{x: 0 \leqslant x \leqslant 2u\}$, that is, of $u+I_u$, where $I_u = \{x: -u \leqslant x \leqslant u\}$. Thus I_u is a neighbourhood of the origin, and therefore X is an A^*-space. Moreover, u is an order unit, by Theorem 1.

For A^*-spaces, the two notions of boundedness coincide: for, if B is a topologically bounded set in such a space, then, for some $\rho > 0$, ρB is contained in an interval of the type $\{x: -u \leqslant x \leqslant u\}$, so that $-\rho^{-1}u \leqslant x \leqslant \rho^{-1}u$ for all $x \in B$. This property is not peculiar to A^*-spaces: for in any locally convex space, a set is bounded if it is weakly bounded; hence, if we replace the topology of an A^*-space by the corresponding weak topology, the resulting A-space has the property that all topologically bounded sets are bounded in the sense of order. However, the property does characterize A^*-spaces among the locally bounded A-spaces, as the following theorem shows. (A topological vector space is 'locally bounded' if it has a neighbourhood of the origin which is topologically bounded.)

THEOREM 3. *An A-space is an A^*-space if and only if it has a neighbourhood of the origin which is bounded in the sense of order.*

Proof. An A^*-space is certainly an A-space and has a neighbourhood of the origin which is bounded in the sense of order. On the other hand, suppose that X is an A-space in which the set $\{x: u \leqslant x \leqslant v\}$ is a neighbourhood of the origin. This set is contained in the symmetric interval $\{x: u-v \leqslant x \leqslant v-u\}$, which is therefore a neighbourhood of the origin. Thus X is an A^*-space.

A locally convex space is locally bounded if and only if its topology is given by a semi-norm (a norm in the case of a Hausdorff space). Thus any A^*-space has its topology given by a semi-norm (indeed, if u is an order unit, we can obtain such a semi-norm by defining $\|x\|$ to be the lower bound of the positive numbers ξ for which $-\xi u \leqslant x \leqslant \xi u$).

Homomorphisms

When X and Y are partially ordered vector spaces, a linear transformation of X into Y is called a *homomorphism* if it is order-preserving: that is to say, if it maps the positive cone of X into the positive cone of Y. We make a few observations on homomorphisms of A-spaces.

THEOREM 4. *Let $\mathscr{F}$ be a class of homomorphisms of an A^*-space X into an A-space Y and suppose that, for some order unit u in X, the set of all points $f(u)$, f belonging to $\mathscr{F}$, is topologically bounded in Y. Then the members of $\mathscr{F}$ are equicontinuous. In particular, every homomorphism of an A^*-space into an A-space is continuous.*

Proof. Let U be a neighbourhood of the origin which is an interval in Y and let $U_{\mathscr{F}}$ be the intersection of all the sets $f^{-1}(U)$, f belonging to $\mathscr{F}$. The fact that the set of points $f(u)$ is bounded implies that $U_{\mathscr{F}}$ absorbs u and $-u$. But, since each f in $\mathscr{F}$ is order-preserving, $U_{\mathscr{F}}$ is an interval in X; hence $U_{\mathscr{F}}$ absorbs the interval $\{x: -u \leqslant x \leqslant u\}$. Thus $U_{\mathscr{F}}$ is a neighbourhood of the origin in X: that is, the members of $\mathscr{F}$ are equicontinuous (at the origin, and hence everywhere by virtue of linearity).

If $\{f_n\}$ is a sequence of homomorphisms of an A^*-space X into an A-space Y, and the sequence $\{f_n(x)\}$ converges for each $x \in X$, the conditions of Theorem 4 are satisfied on taking $\mathscr{F}$ to consist of all the terms f_n. By an argument of the kind used in proving the Banach–Steinhaus theorem [cf. **(8)**], it follows that the limiting transformation f, defined by $f(x) = \lim_{n\to\infty} f_n(x)$, is continuous (it is a homomorphism if the positive cone in Y is closed). From this we deduce that a convergent sequence of linear transformations of X into Y has a continuous limit if it is monotonic (for then $f_1 - f_n$, or $f_n - f_1$, is a homomorphism for each n).

Positive linear functionals

In the theory of partially ordered vector spaces and in certain applications of this theory (**7**, **9**, **10**), the following result is of some importance: if the ordering is strict ($x = 0$ when $0 \leqslant x \leqslant 0$) and if there is an order unit, then there is at least one linear functional which is positive (non-negative and non-zero). From our present point of view, this is very easy to prove: it is implied by the following theorem, when Theorem 1 is taken into account.

THEOREM 5. *There is a positive linear functional on an A^*-space if and only if the origin is not an order unit.*

Proof. Let I be the interior of the positive cone in an A^*-space. By Theorem 2, I is not empty. Hence the positive cone, being convex, is contained in $\bar{I}$, the closure of I. Now the origin is not in I if it is not an order unit, so that in this case, by the 'geometrical form' of the Hahn–Banach theorem [(**6**) 69], the origin lies in a closed hyperplane which does not meet I. Such a hyperplane has an equation $\varphi(x) = 0$, where φ is a continuous linear functional. The convexity of I implies that $\varphi(u)$ has the same sign for all u in I; we may therefore choose φ so that $\varphi(u) > 0$ when $u \in I$. The continuity of φ implies that $\varphi(u) \geqslant 0$ when $u \in \bar{I}$; thus φ is a positive linear functional.

On the other hand, the existence of a positive linear functional φ on an A^*-space implies that the origin is not an order unit. For, the positive cone is then contained in the set $\{x: \varphi(x) \geqslant 0\}$, the interior of which is the set $\{x: \varphi(x) > 0\}$ since φ is continuous (by Theorem 4); thus the interior of the positive cone does not contain the origin.

The conditions of Theorem 5 can easily be satisfied without strict ordering. For example, let X be a vector space of more than one dimension, and let φ be a non-zero linear functional on X: define the positive cone to be the set $\{x: \varphi(x) \geqslant 0\}$, and let the topology be the smallest for which φ is continuous. This prescription gives an A^*-space on which φ is a positive linear functional. The ordering is not strict since $0 \leqslant x \leqslant 0$ when $\varphi(x) = 0$.

The existence of a positive linear functional does not imply the existence of an order unit. It is clear, for example, that a partially ordered vector space X has a positive linear functional if there is a homomorphism of X on to a space that satisfies the conditions of Theorem 5; and this can occur when X has no order unit [cf. (**10**), where some interesting cases are discussed]. Also, it follows from the result on the decomposition of continuous linear functionals into non-negative components, mentioned earlier, that X has an abundance of positive linear functionals if it has a topology with which it is a locally convex Hausdorff A-space; for then every non-zero continuous linear functional is the difference of two linear functionals of which one at least is positive. A necessary and sufficient condition for the existence of a positive linear functional is given in (**11**).

Critical hyperplanes

Under the conditions of Theorem 5, suppose that u is an order unit and let P_u be the set of all positive linear functionals φ such that $\varphi(u) = 1$. Evidently, P_u is a convex set in the adjoint space, and is 'weak*-closed' (closed with respect to the weak* topology). The functionals that comprise P_u are equicontinuous, for, if $\varphi \in P_u$ and $-u \leqslant x \leqslant u$, then $-1 \leqslant \varphi(x) \leqslant 1$; hence P_u is weak*-compact. It follows [(**6**) 83] that every hyperplane of support of P_u which is weak*-closed contains an extreme point of P_u; and, by the Krein–Milman theorem [(**6**) 84], P_u is the weak*-closed convex hull of its extreme points. The hyperplanes $\varphi_0(x) = 0$ corresponding to the extreme points φ_0 of P_u may be called the *critical hyperplanes*; it is easily seen that these do not depend on the particular choice of u. When the ordering is strict, the critical hyperplanes are precisely the 'extreme maximal

ideals' of the given partially ordered vector space (**7, 9, 12**). We consider some of their geometrical properties (assuming only the conditions of Theorem 5).

Let x_0 be a non-zero frontier point of the positive cone. Since the origin is not an order unit, the subspace generated by x_0 does not meet I, the interior of the positive cone (for I is itself a cone). Hence, as in the proof of Theorem 5, this subspace is contained in a closed hyperplane which does not meet I. Such a hyperplane has an equation $\varphi(x) = 0$, where we can suppose that $\varphi \in P_u$; and, since x_0 belongs to this hyperplane, $\varphi(x_0) = 0$. Now the set Φ_0 of all continuous linear functionals φ such that $\varphi(x_0) = 0$ is a weak*-closed hyperplane in the adjoint space. We have just seen that Φ_0 contains a point of P_u; and, since x_0 is a closure point of the positive cone, $\varphi(x_0) \geqslant 0$ for all φ in P_u. Thus Φ_0 is a hyperplane of support of P_u; it therefore contains an extreme point of P_u, say φ_0. Since $\varphi_0(x_0) = 0$, we have shown that *through every non-zero frontier point of the positive cone there passes a critical hyperplane.*

If the space has only one dimension, the positive cone has no non-zero frontier points, and there is only one critical hyperplane (consisting of the origin only). If the space has more than one dimension, let x_0 be any non-zero point outside I. Since I is an open set, it is not contained in the subspace generated by x_0, and therefore has a point u_0 such that the straight segment joining u_0 and x_0 does not contain the origin. This segment must contain a non-zero frontier point x' of the positive cone. Let a critical hyperplane through x' have the equation $\varphi_0(x) = 0$, where $\varphi_0 \in P_u$. Since $\varphi_0(u_0) > 0$ and $x' = \lambda u_0 + (1-\lambda)x_0$, where $0 \leqslant \lambda < 1$, we must have $\varphi_0(x') \leqslant 0$; whereas, of course, $\varphi(u) > 0$. Thus a critical hyperplane through x' separates x_0 from I. We conclude that, under the conditions of Theorem 5, the critical hyperplanes envelop I: that is to say, *a point is an order unit if and only if it lies on the 'positive side' of every critical hyperplane.*

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'NETRICS' OF SETS AND LINEAR SPACES

By K. A. H. GRAVETT (*Oxford*)

[Received 3 September 1957]

Introduction

THE concept of a non-Archimedean metric is generalized after Conrad [*American J. of Math.* 71 (1953) 1–29] and structure-theorems are proved for sets and linear spaces with such a generalized non-Archimedean metric. Conrad's main embedding-theorem is a consequence of the results obtained.

Preliminaries

ø denotes the empty set. If A and B are sets, then $|A|$ denotes the cardinal number of A; $A \setminus B$ denotes the set of elements of A that do not belong to B; and, if x and y are maps defined on A and B respectively and $C \subseteq A \cap B$, then $x = y \mid C$ denotes that, for each c of C, $x(c) = y(c)$.

Let Δ be a partially ordered set. Denote by T the set of $t \subseteq \Delta$ such that, for each δ_1, δ_2 in t, if $\delta_1 \leqslant \delta_2$, then $\delta_1 = \delta_2$. If $s \subseteq \Delta$, $\delta \in \Delta$ and (R) is a binary relation of Δ, put $\delta(R)s$, $s(R)\delta$ if there exists δ' in s with $\delta(R)\delta'$, $\delta'(R)\delta$. Further $\delta(\not R)s$, $s(\not R)\delta$ are taken to mean not-$\delta(R)s$ and not-$s(R)\delta$ respectively, and *not* δ(not-R)s and s(not-R)δ. For each δ in Δ, $[(R)\delta]$ denotes the set of δ' in Δ with $\delta'(R)\delta$. If $s \subseteq \Delta$, denote by $m(s)$ the set of $\delta \in s$ such that, if $\delta' \in s$, then $\delta' \not> \delta$; if, for each $\delta \in s$, $\delta \leqslant m(s)$, put $b(s) = m(s)$; $s \subseteq \Delta$ is said to be *well-ordered* (more usually, *dually* well-ordered) if, for each $s' \subseteq s$, $b(s')$ exists (equivalently, there does not exist an infinite ascending sequence in s).

Let K be a field. Then K^* denotes the set of non-zero elements of K. All linear spaces considered are over the field K.

Netric sets

Let E be a set. A map d of $E \times E$ into T is said to be a *netric* of E (and then E, more precisely $\{E, d\}$, is said to be a *netric set*) if, for each x, y, z in E and δ in Δ,

(i) $d(x, y) = \text{ø}$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) if $\delta \not\leqslant d(x, y) \cup d(y, z)$, then $\delta \not\leqslant d(x, z)$.

It follows that

(iv) if $\delta \not< d(x,y) \cup d(y,z)$, then $d \not< d(x,z)$.†

For suppose that $\delta < \delta' \in d(x,z)$. Then from $\delta \not< d(x,y) \cup d(y,z)$ it follows that $\delta' \not\leqslant d(x,y) \cup d(y,z)$ and $\delta' \not\leqslant d(x,z)$, which is a contradiction.

Let $\{E_1, d_1\}$ and $\{E_2, d_2\}$ be netric sets. A map f of E_1 into E_2 is said to be an *isonetry* if, for each x, y in E_1,

$$d_1(x,y) = d_2(f(x), f(y)).$$

It follows that f is one-to-one.

Let $\{E, d\}$ be a netric set. For each x, y in E and δ in Δ, put $x \sim y(\delta)$ if $\delta \not\leqslant d(x,y)$, and $x \approx y(\delta)$ if $\delta \not< d(x,y)$. Here '$\sim(\delta)$' and '$\approx(\delta)$' are equivalence relations of E, and the corresponding sets of equivalence classes of E are denoted by C_δ and C_δ^* respectively. Each equivalence class $X \in C_\delta$ is contained in an equivalence class $X^* \in C_\delta^*$. E is said to be *equi-cardinal* if, for each δ in Δ and Y in C_δ^*, $|\{X | X \in C_\delta, X \subseteq Y\}|$ depends only on δ (and then is denoted by c_δ).‡ The set of cardinal numbers $\{c_\delta | \delta \in \Delta\}$ indexed by Δ is then said to be the *cardinal-set* of E.

Let $\mathscr{L} = \{L_\delta | \delta \in \Delta\}$ be a set of non-empty sets indexed by Δ. $\mathscr{L}$ is then said to be a Δ-*system of sets*. Denote by $D(\mathscr{L}) = D$ the set of maps x of Δ into $\bigcup L_\delta$ ($\delta \in \Delta$) such that, for each δ in Δ, $x(\delta) \in L_\delta$. D is then said to be the *product* of $\mathscr{L}$. For each x, y in D, denote by $N(x,y)$ the set of $\delta \in \Delta$ for which $x(\delta) \neq y(\delta)$ and put $d(x,y) = b(N(x,y))$ if it exists. A subset E of D is said to be *based* if $d(x,y)$ exists for each x, y in E. For example, let $x \in D$ and denote by $W(x)$ the set of $y \in D$ for which $N(x,y)$ is well-ordered. $W(x)$ is a based subset of D. Evidently $\{E, d\}$ is then a netric set. By Zorn's lemma, each based subset of D is contained in at least one *maximal* based subset of D.

LEMMA 1. *Let M be a maximal based subset of D. Then to each x in M, δ in Δ, and l_δ in L_δ there corresponds $y \in M$ with $y(\delta) = l_\delta$ and $x = y\,|\,[> \delta]$.*

Proof. Put $z(\delta) = l_\delta$ and, for each $\delta' \in \Delta$ with $\delta' \neq \delta$, $z(\delta') = x(\delta')$ ($z \in D$). If $z \in M$, put $y = z$, and then y has the required property. If $z \notin M$, there exist y in M and δ' in Δ such that $\delta' \in N(z,y)$ but $\delta' \not\leqslant m(N(z,y))$. Now $\delta' < \delta$, for $\delta' \not< \delta$ gives $x = z\,|\,[> \delta']$, and there exists $\delta'' \in N(z,y)$ with $\delta'' > \delta'$, from which it follows that

$$\delta'' \in N(x,y),$$

$$\delta'' \leqslant m(N(x,y)) \cap [> \delta'] = m(N(z,y)) \cap [> \delta'] \subseteq m(N(z,y)).$$

† (i), (ii), and (iv) do not yield (iii).

‡ If the netric set E has a transitive group of isonetries, then E is equi-cardinal.

Again $z = y\,|\,[\geqslant \delta]$, for $\delta' < \delta \leqslant \delta'' \in N(z,y)$ gives $\delta' < \delta'' \leqslant m(N(z,y))$; compare the proof of $\delta' < \delta$. From $z = x\,|\,[> \delta]$ it follows that y has the required property.

THEOREM 1. *Let M be a maximal based subset of D. Then M is an equi-cardinal netric set with cardinal-set $\{|L_\delta| \,|\, \delta \in \Delta\}$.*

Theorem 1 follows at once from Lemma 1.

THEOREM 2. *For each x in D, $W(x)$ is a maximal based subset of D.*

Proof. Let $y \in D \setminus W(x)$ and denote by Σ the set of $\delta \in N(x,y)$ for which $N(x,y) \cap [> \delta]$ is well-ordered. Σ is well-ordered, for, if $\delta \in \Sigma$, then

$$\Sigma \cap [> \delta] = N(x,y) \cap [> \delta].$$

For each δ in Δ, put $z(\delta) = y(\delta)$ if $\delta \in \Sigma$, and $z(\delta) = x(\delta)$ if $\delta \notin \Sigma$. $z \in W(x)$. $N(y,z) \neq \emptyset$. Suppose that $\delta \in m(N(y,z))$ $(\delta \in \Sigma)$. However, $N(x,y) \cap [> \delta] \subseteq \Sigma$ since, if $\delta' \in N(x,y) \cap [> \delta]$ and $\delta' \notin \Sigma$, then $z(\delta') = x(\delta')$ for $\delta' \notin \Sigma$, and $z(\delta') = y(\delta')$ for $\delta' > \delta \in m(N(y,z))$, and hence $x(\delta') = y(\delta')$, which is contrary to $\delta' \in N(x,y)$, $N(x,y) \cap [> \delta]$ is well-ordered, and $\delta \in \Sigma$. It follows that $m(N(y,z)) = \emptyset$, and hence $\{W(x), y\}$ is not a based subset of D.

THEOREM 3. *Let M be a maximal based subset of D, let $\Sigma \subseteq \Delta$ such that, if $\Sigma \leqslant \delta \in \Delta$, then $\delta \in \Sigma$, let ϕ be a map of Σ into $\bigcup L_\delta$ $(\delta \in \Delta)$ such that, for each δ in Σ, $\phi(\delta) \in L_\delta$, and let there correspond, to each δ in Σ, $x \in M$ with $x = \phi\,|\,[\geqslant \delta]$. Then there exists $x_0 \in M$ such that $x_0 = \phi\,|\,\Sigma$.*

Proof. Denote by X the set of $x \in D$ with $x = \phi\,|\,\Sigma$. Evidently $X \neq \emptyset$.

For each $x \in X$, denote by $G(x)$ the set of $\delta \in \Delta$ such that, for each y in M, if $\delta \leqslant \delta' \in N(x,y)$, then $\delta' \leqslant m(N(x,y))$. If $x_1 \in M$, $\delta_1 \in \Delta$, and $x = x_1\,|\,[\geqslant \delta_1]$, then $\delta_1 \in G(x)$ since, if $\delta_1 \leqslant \delta' \in N(x,y)$, then

$$\delta_1 \leqslant N(x_1,y), \qquad \delta_1 \leqslant m(N(x_1,y)) \cap [\geqslant \delta_1] = m(N(x,y)) \cap [\geqslant \delta_1].$$

Thus $G(x) \supseteq \Sigma$.

It is enough to prove that there exists $x_0 \in X$ with $G(x_0) = \Delta$.

For each x_1, x_2 in X, put $x_1 \leqslant x_2$ if $G(x_1) \subseteq G(x_2)$ and $x_1 = x_2\,|\,G(x_1)$. The binary relation '$\leqslant$' of X is reflexive, transitive, and inductive. By Zorn's lemma, there exists a maximal element, x_0 say, of X.

A contradiction is derived by assuming that $G(x_0) \neq \Delta$, i.e. there exists $y \in M$ and $\delta_0 \in N(x_0,y)$ with $\delta_0 \nleqslant m(N(x_0,y))$. Denote by x_1 the element of X such that, for each δ in $G(x_0)$, $x_1(\delta) = x_0(\delta)$ and, for each δ not in $G(x_0)$, $x_1(\delta) = y(\delta)$. For each δ in $G(x_0)$ with $\delta \geqslant \delta_0$, $x_0(\delta) = y(\delta)$, since $\delta \in N(x_0,y)$ gives $\delta_0 \leqslant \delta \leqslant m(N(x,y))$. Hence, for *every* δ in Δ

with $\delta \geqslant \delta_0$, $x_1(\delta) = y(\delta)$. It follows that $\delta_0 \in G(x_1)$ and hence that $x_1 > x$ which is a contradiction.

THEOREM 4. *Let M be a maximal based subset of D, let $\Sigma \subseteq \Delta$ such that, if $\Sigma \leqslant \delta \in \Delta$, then $\delta \in \Sigma$, let ϕ be a map of Σ into $\bigcup L_\delta$ $(\delta \in \Delta)$ such that, for each δ in Σ, $\phi(\delta) \in L_\delta$ and let there correspond, to each δ in Σ, $x \in M$ with $x = \phi \,|\, [> \delta]$. Then there exists $x_0 \in M$ such that $x_0 = \phi \,|\, \Sigma$.*

Proof. Let $\delta \in \Sigma$ and $x \in M$ with $x = \phi \,|\, [> \delta]$. By Lemma 1, there exists $x' \in M$ with $x' = \phi \,|\, [\geqslant \delta]$. An application of Theorem 3 completes the proof.

Netric linear spaces

Let L be a linear space. A map d of L into T is said to be a *netric* of the *linear space* L if, for each x, y in L, for k in K^*, and δ in Δ,

(i) $d(x) = \mathfrak{o}$ if and only if $x = 0$;

(ii) $d(kx) = d(x)$;

(iii) if $\delta \not\leqslant d(x) \cup d(y)$, then $\delta \not\leqslant d(x+y)$.

It follows that, if $\delta \lessdot d(x) \cup d(y)$, then $\delta \lessdot d(x+y)$. For each x, y in L put $d^*(x, y) = d(x-y)$; d^* is a netric of the *set* L. There is a natural one-to-one correspondence between netrics of the linear space L and an easily characterized class of netrics of the set L. Definitions and results concerning netric sets are applied to netric linear spaces considered as netric sets.

Let $\{L_1, d_1\}$ and $\{L_2, d_2\}$ be netric linear spaces. A map f of L_1 into L_2 is said to be an *isonetry* if f is an isomorphism and an isonetry of L_1 into L_2 considered as netric sets: equivalently, for each x in L, $d_1(x) = d_2(f(x))$.

Let $\{L, d\}$ be a netric linear space. For each δ in Δ, denote by J_δ the set of $x \in L$ with $\delta \not\leqslant d(x)$ and by J^δ the set of $x \in L$ with $\delta \lessdot d(x)$. J_δ $(\in C_\delta$ of $\{L, d^*\})$ and J^δ $(\in C^*_\delta)$ are linear subspaces of L and $J_\delta \subseteq J^\delta$. Put $L_\delta = J^\delta / J_\delta$. L (i.e. $\{L, d^*\}$) is equi-cardinal with cardinal-set $\{|\, L_\delta \,||\, \delta \in \Delta\}$.

Let $\mathscr{L} = \{L_\delta \,|\, \delta \in \Delta\}$ be a set of linear spaces indexed by Δ. $\mathscr{L}$ is then said to be a Δ-*system of linear spaces*. $\mathscr{L}$ is also a Δ-system of sets. With respect to component-wise compositions, the product $D(\mathscr{L}) = D$ is a linear space. For each x in D, put $N(x) = N(x, 0)$ and $d(x) = d(x, 0)$ if it exists. A linear subspace L of D is said to be *based* if it is a based subset of D. Evidently $\{L, d\}$ is then a netric linear space (and d^* is the netric of L considered as a based *subset* of D). Each based linear subspace of D is contained in at least one *maximal* based

linear subspace of D. A maximal based linear subspace of D is, in fact, a maximal based *subset* of D. For let $y \in D$ and suppose that $d(x, y)$ exists for each $x \in M$. Then $d(x-y)$, $d(-kx+ky)$, and $d(x+ky)$ exist for each x in M and k in K. Denote $W(0)$ by W (i.e. W is the set of $x \in D$ for which $N(x)$ is well-ordered) and denote by F the set of $x \in D$ for which $N(x)$ is a finite subset of Δ. W is a maximal based linear subspace of D, and F is a linear subspace of W.

Embeddings

Let $\{E_1, d_1\}$ and $\{E_2, d_2\}$ be netric sets. E_2 is said to be an *extension* of E_1 if $E_2 \supseteq E_1$ and $d_1 = d_2 \mid E_1 \times E_1$. E_2 is said to be an *immediate* extension of E_1 if, further, to each y in $E_2 \setminus E_1$, x in E_1, and δ in $d(x, y)$ there corresponds $x' \in E_1$ with $\delta \not\leqslant d(x', y)$.

THEOREM 5. *Let $\{E, d\}$ be an equi-cardinal netric set with cardinal-set $\{c_\delta \mid \delta \in \Delta\}$ and let $\mathscr{L} = \{L_\delta \mid \delta \in \Delta\}$ be a Δ-system of sets with $|L_\delta| = c_\delta$ for each δ in Δ. Then there is an isonetry f of E onto a based subset of the product $D(\mathscr{L})$ such that each maximal based subset of $D(\mathscr{L})$ containing $f(E)$ is an immediate extension of $f(E)$.*

Proof. For each δ in Δ, let f_δ be a map of C_δ onto L_δ such that, for each Y in C_δ^*,

$$f_\delta \mid \{X \mid X \in C_\delta, X \subseteq Y\}$$

is one-to-one and onto L_δ. For each x in E and δ in Δ, put

$$(f(x))(\delta) = f_\delta(X),$$

where $x \in X \in C_\delta$. Now f is an isonetry of E onto a based subset of $D(\mathscr{L})$. This is a consequence of the following two observations. Let $x, y \in E$. If $\delta \in d(x, y)$, then $x \approx y(\delta)$ but $x \not\sim y(\delta)$, and hence

$$(f(x))(\delta) \neq (f(y))(\delta).$$

If $\delta \in \Delta$ and

$$(f(x))(\delta) \neq (f(y))(\delta),$$

then $x \not\sim y(\delta)$, and hence $\delta \leqslant d(x, y)$. Let M be a maximal based subset of $D(\mathscr{L})$ containing $f(E)$. M is an immediate extension of $f(E)$. For let $y \in M \setminus f(E)$, $x \in E$, and $\delta \in d(f(x), y)$. Let $x' \in E$ with $x \approx x'(\delta)$ and $f_\delta(X') = y(\delta)$, where $x' \in X' \in C_\delta$. Then $\delta \not< d(f(x'), y)$ since $\delta \not< d(f(x), f(x'))$ and $\delta \notin N(f(x'), y)$ since $(f(x'))(\delta) = y(\delta)$. Thus $\delta \not\leqslant d(f(x'), y)$.

The method of proof of Theorem 5 yields the following result. Let $\{E, d\}$ be a netric set. Then there is an isonetry f of E onto a based subset of the product $D(\mathscr{L})$ for suitable choice of Δ-system of sets $\mathscr{L}$.

THEOREM 6. *Let $\{L,d\}$ be a netric linear space. Then there is an isonetry f of L onto a based linear subspace of the product $D(\mathscr{L})$, where*

$$\mathscr{L} = \{L_\delta = J^\delta/J_\delta \mid \delta \in \Delta\},$$

such that each maximal based linear subspace of $D(\mathscr{L})$ containing $f(L)$ is an immediate extension of $f(L)$.

The proof of Theorem 5 (with the maps f_δ homomorphisms) yields a proof of Theorem 6.

Completeness

Let $\{E,d\}$ be a netric set. E is said to be *complete* if there does not exist a proper immediate extension of E. An analogous definition applies to netric linear spaces. It is a consequence of later results that a complete netric linear space is complete considered as a netric set.

Let $\{E,d\}$ be a netric set. A subset A of $E\times\Delta$ is said to be an *overlap* of E if, for each (x_i,δ_i) in A and δ in Δ with $\delta \geqslant \delta_i$ $(i=1,2)$, $\delta \not\leqslant d(x_1,x_2)$. Again $x_0 \in E$ is said to be a *limit* of A if, for each (x,δ) in A, $\delta \not\leqslant d(x,x_0)$. E is said to be *overlap-complete* if every overlap of E has a limit in E.

THEOREM 7. *Let $D(\mathscr{L})$ be the product of a Δ-system of sets $\mathscr{L} = \{L_\delta \mid \delta \in \Delta\}$ and let M be a maximal based subset of $D(\mathscr{L})$. Then M is overlap-complete.*

Theorem 7 follows at once from Theorem 3. The completeness underlying Theorem 4 can be formulated for general netric sets.

THEOREM 8. *Let $\{E,d\}$ be a netric set. If E is overlap-complete, then E is complete.*

Proof. Let $\{E',d\}$ be a proper immediate extension of E and let $y \in E' \setminus E$. Denote by A the set of $(x,\delta) \in E\times\Delta$ with $\delta \not\leqslant d(y,x)$. A is an overlap of E. For, if $(x_i,\delta_i) \in A$ and $\delta \in \Delta$ with $\delta \geqslant \delta_i$, then $\delta \not\leqslant d(y,x_i)$ $(i=1,2)$, and hence $\delta \not\leqslant d(x_1,x_2)$. Let $x \in E$ be a limit of A. Let $\delta \in d(x,y)$ and $x' \in E$ with $\delta \not\leqslant d(x',y)$. Then $(x',\delta) \in A$, and hence $\delta \not\leqslant d(x,x')$, which gives a contradiction to $\delta \not\leqslant d(x',y)$ and $\delta \in d(x,y)$.

THEOREM 9. *Let $\{E,d\}$ be an equi-cardinal netric set with cardinal-set $\{c_\delta \mid \delta \in \Delta\}$ and let $\mathscr{L} = \{L_\delta \mid \delta \in \Delta\}$ be a Δ-system of sets with $|L_\delta| = c_\delta$ for each δ in Δ. Then E is complete if and only if E is isonetric to a maximal based subset of $D(\mathscr{L})$.*

COROLLARY. *An equi-cardinal netric set is complete if and only if it is overlap-complete.*

Analogous results hold for netric linear spaces with, in the corollary, completeness of the netric linear space considered either as *netric set* or as *netric linear space*. Theorem 9, its corollary, and their analogues are immediate consequences of preceding results.

Maximal based subsets

Let $D(\mathscr{L}) = D$ be the product of a Δ-system of sets $\mathscr{L} = \{L_\delta \mid \delta \in \Delta\}$. Let M_1 and M_2 be maximal based subsets of D and $y_1 \in M_1$ and $y_2 \in M_2$ be arbitrary but fixed.

THEOREM 10. *There is an isonetry f of M_1 onto M_2 such that $f(y_1) = y_2$.*

COROLLARY 1. *An overlap-complete equi-cardinal netric set is determined up to isonetry by its cardinal-set.*

COROLLARY 2. *Every maximal based subset of D has a transitive group of isonetries.*

Theorem 10 follows from the observations below together with Zorn's lemma. Corollary 1 is an immediate consequence of the theorem together with earlier results. Corollary 2 is an immediate consequence of the theorem.

For each δ in Δ, let '$+_i$' be an arbitrary, but fixed, group-composition† of L for $i = 1, 2$ with $\langle\delta\rangle$ an isomorphism of $\{L_\delta, +_1\}$ onto $\{L_\delta, +_2\}$. Let $y_i \in N_i \subseteq M_i$ $(i = 1, 2)$ with f an isonetry of N_1 onto N_2 such that

(i) $f(y_1) = y_2$;

(ii) to each $x, x' \in N_1$ and $\delta \in d(x, x')$,

$$\langle\delta\rangle(x(\delta) -_1 x'(\delta)) = (f(x))(\delta) -_2 (f(x'))(\delta);$$

it then follows that the equation holds for each x, x' in N_1 and δ in Δ with $\delta \not< d(x, x')$.

Let $z_1 \in M_1 \setminus N_1$. Denote by Σ the set of $\delta \in \Delta$ to which there corresponds $x \in N_1$ with $\delta \not< d(x, z_1)$ and put

$$\phi(\delta) = \langle\delta\rangle(z_1(\delta) -_1 x(\delta)) +_2 (f(x))(\delta).$$

If $x' \in N_1$ with $\delta \not< d(x', z_1)$, then

$$\delta \not< d(x, x'), \quad -_2\langle\delta\rangle(x(\delta) +_2 (f(x))(\delta) = -_2\langle\delta\rangle(x'(\delta)) +_2 (f(x'))(\delta),$$

and hence ϕ is well-defined. If $\delta \leqslant \delta' \in \Delta$, then $\delta' \in \Sigma$.

Let $\delta, \delta' \in \Sigma$ and $x \in N_1$ with $\delta' > \delta \not< d(x, z_1)$.

$$\phi(\delta') = \langle\delta'\rangle(z_1(\delta') -_1 x(\delta')) +_2 (f(x))(\delta') = (f(x))(\delta')$$

† Theorem 10 can be proved without introducing group-compositions. The advantage of the given proof is its applicability to the proof of Theorem 11.

since $\delta' \not\leqslant d(x, z_1)$, and hence $\phi = f(x) \mid [> \delta]$. By Theorem 4 there exists $z_2 \in M_2$ such that $z_2 = \phi \mid \Sigma$.

For each x in N_1 and δ in $d(x, z_1)$,

$$(f(x))(\delta) -_2 z_2(\delta) = (f(x))(\delta) -_2 \phi(\delta)$$
$$= (f(x))(\delta) -_2 [\langle\delta\rangle(z_1(\delta) -_1 x(\delta)) +_2 (f(x))(\delta)] = \langle\delta\rangle(x(\delta) -_1 z_1(\delta)).$$

It follows that $z_2 \notin N_2$.

Finally it is proved that, for each x_1 in N_1, $d(x_1, z_1) = d(f(x_1), z_1)$. If $\delta \in d(x_1, z_1)$, then

$$\delta \ll d(x_1, z_1), \qquad z_2(\delta) \neq (f(x_1))(\delta)$$

since $z_1(\delta) \neq x_1(\delta)$, and $\delta \not\leqslant d(f(x_1), z_2)$. Again, if $\delta \in d(f(x_1), z_2)$, then $\delta \leqslant d(x_1, z_1)$, for, if $\delta \not\leqslant d(x_1, z_1)$, then $z_2(\delta) = (f(x_1))(\delta)$.

Maximal based linear subspaces

Let $D(\mathscr{L}) = D$ be the product of a Δ-system of linear spaces $\mathscr{L} = \{L_\delta \mid \delta \in \Delta\}$. Let M_1 and M_2 be maximal based linear subspaces of D.

THEOREM 11. *There is an isonetry f of M_1 onto M_2 such that, for each x_1 in M_1 and δ in $d(x)$, $x(\delta) = (f(x))(\delta)$.*

COROLLARY. *An overlap-complete netric linear space is determined up to isonetry by its cardinal-set.*

Theorem 11 follows from the observations below together with Zorn's lemma.† Its corollary is an immediate consequence of the theorem together with earlier results.

The observations constituting the proof of Theorem 10 are adapted as follows: For each δ in Δ, '$+_1$' and '$+_2$' are each the given group-composition of L_δ and $\langle\delta\rangle$ is the identity isomorphism. $y_1 = y_2 = 0$. Let N_i be a linear subspace of M_i $(i = 1, 2)$ and let f be an isonetry of N_1 onto N_2 such that, for each x in N_1 and δ in $d(x)$, $x(\delta) = (f(x))(\delta)$. Equivalently, for each x, x' in N_1 and δ in $d(x - x')$,

$$\langle\delta\rangle(x(\delta) -_1 x'(\delta)) = (x - x')(\delta) = (f(x - x')(\delta)) = (f(x))(\delta) -_2 (f(x'))(\delta).$$

z_1, Σ, ϕ, and z are defined as before.

For each x in N_1 and k in K^*,

$$d(x + kz_1) = d(-k^{-1}x - z_1) = d(-k^{-1}f(x) - z_2) = d(f(x) + kz_2)$$

and, if $\delta \in d(x + kz_1)$ $(= d(-k^{-1}x - z_1))$, then

$$(x + kz_1)(\delta) = -k[(-k^{-1}x - z_1)(\delta)]$$
$$= -k[(-f(k^{-1}x) - z_2)(\delta)] = (f(x) + kz_2)(\delta).$$

† Alternatively Theorem 11 can be proved by constructing an isonetry f_i of F into L_i such that, for each x in F and δ in $d(x)$, $x(\delta) = (f_i(x))(\delta)$ $(i = 1, 2)$ and then showing that there is an isonetry of M_1 onto M_2 over the isonetry $f_2 f_1^{-1}$ of F_1 onto F_2.

THE GROUPS $\pi_r(V_{n,m})$ (III)

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Introduction

THIS is the third of a sequence of five papers, the previous ones being (8) and (9), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those groups which are given in the following tables. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 1$. A full table of results can be found in (8) 249. Some of these, namely the values of $\pi^p_{k,m}$ for $2 \leqslant p \leqslant 5$ and $k \geqslant p+2$, have also been obtained, independently and by different methods, by Yoshihiro Saito (10).

For the notation used throughout the body of this paper please see (8), especially §§ 1, 2, and 3.1. Also please note that sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,4}$

k	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
1	Z_∞	Z_2	Z_2	0	Z_∞	0
3	Z_2	Z_2	Z_4+Z_∞	$Z_2+Z_2+Z_2$		
4	Z_2+Z_2	$Z_\infty+Z_{12}+Z_4+Z_\infty$	$Z_2+Z_2+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2+Z_2$		
5	Z_8	Z_2+Z_2	Z_2+Z_2	Z_2		
6	0	$Z_{12}+Z_\infty$	Z_2	$Z_\infty+Z_4$		
$4s-1$	Z_2	Z_2+Z_2	Z_2	Z_2+Z_2		
$4s+1$	Z_8	Z_2+Z_2	Z_2	Z_2		
$4s$	Z_2+Z_2	$Z_{24}+Z_4+Z_\infty$	$Z_2+Z_2+Z_2$	Z_2+Z_2		
$4s+2$	0	$Z_{12}+Z_\infty$	Z_2	Z_4		

TABLE FOR $\pi^p_{2,3}$

$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
Z_∞	Z_2	Z_2	0	Z_∞	0

8. Calculation of $\pi^p_{k,4}$

We consider the fibring $V_{k+4,4}/V_{k+3,3} \to S^{k+3}$ and examine the sequence

$$(C) \qquad \to \pi_{k+p+1}(S^{k+3}) \xrightarrow{\Delta_*} \pi^p_{k,3} \xrightarrow{i_{k+p*}} \pi^p_{k,4} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+3}) \to.$$

8.1. $k \equiv 0 \pmod 4$.

In this case there is a three-field on S^{k+3} (**5**, **12**), and so the fibring admits a cross-section p. Hence Theorem 1.1 gives

$$\pi^p_{k,4} = i_*\pi^p_{k,3} + \mathrm{p}_*\pi_{k+p}(S^{k+3}).$$

Using the values of $\pi^p_{k,3}$ as calculated in § 7.31, we obtain the values shown in the table for $\pi^p_{k,4}$ when $k \equiv 0 \pmod 4$.

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+4,4}\} = 0 \quad \text{for } k \equiv 0 \pmod 4.$$

8.2. $k \equiv 2 \pmod 4$ *and* $\geqslant 6$.

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+3*}} \pi^2_{k,4} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+3}) \to$$

i.e.

$$\to Z_\infty \to Z_2 \to \pi^2_{k,4} \to 0,$$

by § 7.32 (*b*). But $i^{-1}_{k+2*}(0) \neq 0$ since otherwise there would be a cross-section in the above fibring, by Theorem 1.2, and so a three-field on S^{k+3}, which is impossible (**5**, **12**). Hence $i^{-1}_{k+2*}(0) = \pi^2_{k,3}$, whence we have that

$$\pi^2_{k,4} = 0.$$

Note that Δ_* is onto, whence the image of p_{k+3*} is the Z_∞ subgroup generated by $2\{h_{k+3,k+3}\}$. Further we have from Corollary 1.5 that $\{t_{k+4,4}\}$ generates $i^{-1}_{k+2*}(0)$. Hence, by § 7.32 (*b*),

$$\{t_{k+4,4}\} = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}.$$

(*b*) When $p = 3$, (C) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_*} \pi^3_{k,3} \xrightarrow{i_{k+3*}} \pi^3_{k,4} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \to,$$

i.e.

$$\to Z_2 \to Z_{12} + Z_2 \to \pi^3_{k,4} \to Z_\infty \to 0,$$

by § 7.32 (*c*) and since the image of p_{k+3*} is a Z_∞ subgroup by (*a*). Also

$$i^{-1}_{k+3*}(0) = t_{k+4,4*}\,\pi_{k+3}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+3}\{t_{k+4,4}\} = h^*_{k+2,k+3}\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\} = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0)$ is the Z_2 summand of $\pi^3_{k,3}$, whence

$$\pi^3_{k,4} = Z_{12} + Z_\infty,$$

generated by $\{i_{k+1,3}h_{k,k+3}\}$ and a such that

$$p_{k+4,1*}a = 2\{h_{k+3,k+3}\}.$$

Note that $\Delta^{-1}_*(0) = 0$, and so that p_{k+4*} is trivial.

(*c*) When $p = 4$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \to,$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^4_{k,4} \to 0,$$

by § 7.32 (*d*) and since p_{k+4*} is trivial by (*b*). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{i_{k+2,1}\, \mathrm{p} h_{k+1,k+2}\} = i_{k+4*}\, \mathrm{p}_* 12\{h_{k+1,k+4}\}$$
$$\in i_{k+4*}\, 2\mathrm{p}_*\, \pi_{k+4}(S^{k+1}) = 0, \quad \text{by 7.32 } (d).$$

Hence $i^{-1}_{k+4*}(0) = 0$, and so we have that

$$\pi^4_{k,4} = Z_2,$$

generated by $\{i_{k+2,2}\, \mathrm{p} h_{k+1,k+4}\}$. Note that thus Δ_* is trivial, whence p_{k+5*} is onto.

(*d*) When $p = 5$ and $k \geqslant 10$, (C) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+3}) \xrightarrow{\Delta_*} \pi^5_{k,3} \xrightarrow{i_{k+5*}} \pi^5_{k,4} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \to,$$

i.e.
$$\to Z_{24} \to Z_2 \to \pi^5_{k,4} \to Z_2 \to 0,$$

by § 7.32 (*e*) and since p_{k+5*} is onto by (*c*). Also

$$i^{-1}_{k+5*}(0) = t_{k+4,4*}\pi_{k+5}(S^{k+2}) = \{i_{k+2,1}\, \mathrm{p} h_{k+1,k+2}\}_* \pi_{k+5}(S^{k+2})$$
$$\subset i_{k+2,1*}\, \mathrm{p}_*\, \pi_{k+5}(S^{k+1}) = 0.$$

Hence $\pi^5_{k,4}$ has four elements. But, by Theorem 4.2 (*b*),

$$\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_4,$$

by § 7.2 (*e*). Thus $\pi^5_{k,4} = Z_4$ when $k \geqslant 10$

and is generated by an element a such that $p_{k+4,1*}\, a = \{h_{k+3,k+5}\}$. Note that Δ_* is trivial, whence p_{k+6*} is onto.

(*e*) When $p = 5$ and $k = 6$, (C) gives

$$\xrightarrow{p_{12*}} \pi_{12}(S^9) \xrightarrow{\Delta_*} \pi^5_{6,3} \xrightarrow{i_{11*}} \pi^5_{6,4} \xrightarrow{p_{11*}} \pi_{11}(S^9) \to,$$

i.e.
$$\to Z_{24} \to Z_\infty + Z_2 \to \pi^5_{6,4} \to Z_2 \to 0,$$

by § 7.32 (*f*) and since p_{11*} is onto by (*c*). Also

$$i^{-1}_{11*}(0) = t_{10,4*}\pi_{11}(S^8) = \{i_{8,1}\mathrm{p} h_{7,8}\}_* \pi_{11}(S^8) \subset i_{8,1*}\, \mathrm{p}_*\, \pi_{11}(S^7) = 0.$$

Hence $\pi^5_{6,4}$ *is an extension of* $Z_\infty + Z_2$ *by* Z_2.

Note that thus Δ_* is trivial, whence p_{12*} is onto.

In determining the structure of $\pi^5_{6,4}$ we first consider the sequence associated with the fibring $V_{10,4}/S^6 \to V_{10,3}$, which is of the form

$$\to \pi_{11}(S^6) \to \pi^5_{6,4} \to \pi^4_{7,3} \to \pi_{10}(S^6) \to,$$

i.e.
$$\to Z_\infty \to \pi^5_{6,4} \to Z_4 \to 0,$$

by § 7.2 (*e*). Bearing in mind the result of the last paragraph, we see that $\pi^5_{6,4}$ is also an extension of Z_∞ by Z_4. Hence

$$\pi^5_{6,4} \textit{ is either } Z_\infty + Z_2 \textit{ or } Z_\infty + Z_4.$$

To determine which, we consider $\pi_{11}(P^9_5) \approx \pi^5_{6,4}$ by Theorem 2.3 (*g*). When $k \geqslant 3$, let C^{k+2} be the space described in § 7.32 (*f*), and let D^{k+3}_λ be the space consisting of C^{k+2} to which a $(k+3)$-cell has been attached by a map ϕ_λ such that

$$\phi_\lambda | \dot{E}^{k+3} \to S^k \vee S^{k+1} \subset C^{k+2},$$

$$\{\delta\phi_\lambda\} = j_*\{h_{k+1,k+2}\} + i_*\lambda\{h_{k,k+2}\}.$$

It was shown in § (E) of the Appendix to (**9**) that, when $k \equiv 2 \pmod 4$ and $\geqslant 6$, $(P^{k+3}_{k-1}, P^{k+2}_{k-1})$ is of the same homotopy type as $(D^{k+3}_\lambda, C^{k+2})$ for some particular $\lambda = 0$ or 1. So consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{11}(C^8) & \xrightarrow{i_{11*}} & \pi_{11}(D^9_\lambda) & \xrightarrow{j_{11*}} & \pi_{11}(D^9_\lambda, C^8) & \xrightarrow{\delta_{11*}} & \pi_{10}(C^8) \to \\
\uparrow \mathfrak{E} & & \uparrow \mathfrak{E}_1 & & \uparrow \mathfrak{E}_2 & & \uparrow \mathfrak{E} \\
\to \pi_{10}(C^7) & \xrightarrow{i'_{10*}} & \pi_{10}(D^8_\lambda) & \xrightarrow{j'_{10*}} & \pi_{10}(D^8_\lambda, C^7) & \xrightarrow{\delta'_{10*}} & \pi_9(C^7) \to .
\end{array}$$

We know from § 7.32 (*f*) that $\pi_{11}(C^8)$ is $Z_\infty + Z_2$, and from above that $\pi_{11}(D^9_\lambda)$ is either $Z_\infty + Z_2$ or $Z_\infty + Z_4$. We first prove that $i^{-1}_{11*}(0) = 0$. Then we shall show that $\pi_{10}(D^8_\lambda)$ is finite, and that it contains an element a such that $j_{11*}\mathfrak{E}_1 a$ is non-zero. Thus $\mathfrak{E}_1 a$ is an element of $\pi_{11}(D^9_\lambda)$ which is non-zero, of finite order, and non-trivial under j_{11*}. Hence $\pi_{11}(D^9_\lambda)$ must be $Z_\infty + Z_4$.

The first assertion follows from the fact that

$$i^{-1}_{11*}(0) = \bar{\imath}_{11*}\{h_{7,8} + \lambda h_{6,8}\}_* \pi_{11}(S^8)$$

by Theorems 1 and 2 of (**14**) (where $\bar{\imath}$ denotes the identical injection of $S^6 \vee S^7$ into C^8), which is of finite order and yet a subgroup of $\bar{\imath}_{11*}\pi_{11}(S^7 \vee S^6)$ which equals Z_∞ by § 7.32 (*f*). Hence $i^{-1}_{11*}(0) = 0$.

Secondly, by § 7.32 (*f*), $\pi_{10}(C^7)$ has eight elements and, by Theorem 1 of (**14**),

$$\pi_{10}(D^8_\lambda, C^7) \approx \pi_9(S^7) = Z_2.$$

Thus $\pi_{10}(D^8_\lambda)$ is of finite order. Further, by Theorems 1 and 2 of (**14**),

$$\delta'_{10*}\pi_{10}(D^8_\lambda, C^7) = \bar{\imath}'_{9*}\{h_{6,7} + \lambda h_{5,7}\}_* \pi_9(S^7),$$

which is generated by

$$i'_{9*}h^*_{7,9}\{h_{6,7}+\lambda h_{5,7}\} = i'_{9*}(12\{h_{6,9}\}+12\lambda h_{5,8*}\{h_{8,9}\}) = i'_{9*}\,12\{h_{6,9}\} = 0,$$

by the argument in § 7.32 (*f*). Thus $\delta'_{10*}(D^8_\lambda, C^7) = 0$, which implies that j'_{10*} is onto. But $\mathfrak{E}_2$ is an isomorphism by Theorem 3.2 (B). Thus $\tau_{10}(D^8_\lambda)$ contains an element a of finite order such that

$$j_{11*}\,\mathfrak{E}_1 a = \mathfrak{E}_2 j''_{10*}\,a \neq 0.$$

This completes the proof, and so we have that

$$\pi^5_{6,4} = Z_\infty + Z_4,$$

generated by $\{i_{7,3}h_{6,11}\}$ and a such that $p_{10,1*}a = \{h_{9,11}\}$ and $4a = 0$.

8.3. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

We first calculate

$$\{t_{k+4,4}\} \in \pi^2_{k,3} = Z_4 + Z_\infty$$

by §§ 7.1 and 5.2 (*c*). We have that

$$t_{k+4,4}|S^{k+1} = i_{k+2,1}\,t_{k+3,3}$$

by § 2.3 (*b*), and that $\{t_{k+3,3}\} = 0$ by § 7.1. Thus we can extend $i_{k+2,1}\,t_{k+3,3}$ over the hemisphere E^{k+2}_+ of S^{k+2}, and, since $t_{k+3,3}$ is a symmetric map (2.3 *a*), we can extend it symmetrically over E^{k+2}_-. Denote this extension by

$$g\colon S^{k+2} \to i_{k+2,1}\,V_{k+2,2} \subset V_{k+3,3}.$$

Now we use construction Q^r of § 6, with

$$r = k+2, \quad X = V_{k+3,3}, \quad f_1 = t_{k+4,4}, \quad f_2 = g.$$

Then we have $\quad 2\{h\} = \{f_1\}+\{f_2\} = \{t_{k+4,4}\}+\{g\}.$

Hence

$$\begin{aligned} p_{k+3,1*}2\{h\} &= p_{k+3,1*}\{t_{k+4,4}\}+p_{k+3,1*}\{g\} \\ &= 2\{h_{k+2,k+2}\}, \end{aligned}$$

by § 2.3 (*b*) and since $\{g\} \in i_*\,\pi^2_{k,2}$. Thus

$$p_{k+3,1*}\{h\} = \{h_{k+2,k+2}\},$$

and

$$\{h\} = \{\mathrm{p}h_{k+2,k+2}\}+i_*w,$$

where $w \in \pi^2_{k,2}$.

Further $p_{k+2,1}\,g\colon S^{k+2} \to S^{k+1}$ is a symmetric map and such that

$$p_{k+2,1}\,g u^{-1}_{k+1} = p_{k+2,1}\,t_{k+3,3}\,u^{-1}_{k+1}\colon P^{k+1} \to S^{k+1},$$

which is essential by § 2.3 (*c*); whence $p_{k+2,1}\,g$ is essential by Theorem 6.1. Thus $\{g\}$ generates the Z_4 summand of $\pi^2_{k,3}$ (§§ 7.1 and 5.2 *c*). Hence

$$\{t_{k+4,4}\} = 2\{h\}-\{g\} = 2\{\mathrm{p}h_{k+2,k+2}\}+2i_*w-\{g\},$$

i.e.

$$\{t_{k+4,4}\} = 2\mathrm{p}_*\{h_{k+2,k+2}\}+i_{k+2,1*}a,$$

where a is a generator of $\pi^2_{k,2}$, i.e.

$$p_{k+2,1*}a = \{h_{k+1,k+2}\}.$$

Thus we have the following:

(a) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+2*}} \pi^2_{k,4} \to \pi_{k+2}(S^{k+3}) \to,$$

i.e.

$$\to Z_\infty \to Z_4 + Z_\infty \to \pi^2_{k,4} \to 0,$$

by §§ 7.1 and 5.2 (c), where $i^{-1}_{k+2*}(0)$ is generated by $\{t_{k+4,4}\}$. Using the above result we have that

$$\pi^2_{k,4} = Z_8,$$

generated by $\{i_{k+3,1}\, \mathrm{p}h_{k+2,k+2}\}$. Note that thus $\Delta^{-1}_*(0) = 0$, whence p_{k+3*} is trivial.

(b) When $p = 3$, (C) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_*} \pi^3_{k,3} \xrightarrow{i_{k+3*}} \pi^3_{k,4} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^3_{k,4} \to 0,$$

by §§ 7.1 and 5.2 (d), and since p_{k+3*} is trivial by (a). Also

$$i^{-1}_{k+3*}(0) = t_{k+4,4*}\pi_{k+3}(S^{k+2}),$$

which is generated by

$$\begin{aligned} h^*_{k+2,k+3}\{t_{k+4,4}\} &= h^*_{k+2,k+3}(i_{k+2,1*}a + 2\mathrm{p}_*\{h_{k+2,k+2}\}) \\ &= h^*_{k+2,k+3}\, i_{k+2,1*}a + 2\mathrm{p}_*\{h_{k+2,k+3}\} = i_{k+2,1*}h^*_{k+2,k+3}a. \end{aligned}$$

But

$$\begin{aligned} p_{k+2,1*}h^*_{k+2,k+3}a &= h^*_{k+2,k+3}p_{k+2,1*}a, \quad \text{by Lemma 3.1 } (b), \\ &= h^*_{k+2,k+3}\{h_{k+1,k+2}\} = \{h_{k+1,k+3}\}. \end{aligned}$$

Hence, by §§ 7.1 and 5.2 (d), $i^{-1}_{k+3*}(0)$ is a Z_2 summand of $\pi^3_{k,3}$. Thus

$$\pi^3_{k,4} = Z_2 + Z_2,$$

generated by $\{i_{k+1,3}h_{k,k+3}\}$ and $\{i_{k+3,1}\, \mathrm{p}h_{k+2,k+3}\}$. Note that thus

$$\Delta^{-1}_*(0) = 0,$$

whence p_{k+4*} is trivial.

(c) When $p = 4$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \to,$$

i.e.

$$\to Z_2 \to \pi^4_{k,3} \to \pi^4_{k,4} \to 0,$$

since p_{k+4*} is trivial by (b). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{t_{k+4,4}\} = h^*_{k+2,k+4}(i_*a + 2\mathrm{p}_*\{h_{k+2,k+2}\})$$
$$= h^*_{k+2,k+4} i_* a + 2\mathrm{p}_*\{h_{k+2,k+4}\} = i_* h^*_{k+2,k+4} a.$$

But
$$p_{k+2,1*} h^*_{k+2,k+4} a = h^*_{k+2,k+4} p_{k+2,1*} a \quad (3.1\,b)$$
$$= h^*_{k+2,k+4}\{h_{k+1,k+2}\} = 12\{h_{k+1,k+4}\}.$$

Hence, by §§ 7.1 and 5.2 (*f*) and (*h*), $i^{-1}_{k+4*}(0)$ is a Z_2 summand of $\pi^4_{k,3}$. Thus we have that

$$\pi^4_{k,4} = Z_2 \text{ for } k \geqslant 9; \quad \pi^4_{5,4} = Z_2 + Z_2,$$

generated respectively by $\{i_{k+3,1}\,\mathrm{p}h_{k+2,k+4}\}$, and $\{i_{6,3}h_{5,9}\}$ and $\{i_{8,1}\,\mathrm{p}h_{7,9}\}$. Note that again p_{k+5*} is trivial.

(*d*) When $p = 5$ and $k \geqslant 9$, we use Theorem 4.2 (*b*) by which $\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_2$ by § 7.32 (*d*). Thus

$$\pi^5_{k,4} = Z_2 \quad \text{when } k \geqslant 9,$$

and is easily seen to be generated by $\{i_{k+3,1}\,\mathrm{p}h_{k+2,k+5}\}$.

(*e*) When $p = 5$ and $k = 5$, we consider the sequence associated with the fibring $V_{9,4}/S^5 \to V_{9,3}$, i.e.

$$\to \pi_{10}(S^5) \xrightarrow{i_{6,3*}} \pi^5_{5,4} \xrightarrow{p_{9,3*}} \pi^4_{6,3} \to \pi_9(S^5) \xrightarrow{i_{6,3*}},$$

i.e.
$$Z_2 \to \pi^5_{5,4} \to Z_2 \to Z_2 \to Z_2 + Z_2 \to$$

by § 7.32 (*d*) and (*c*) above. But

(i) $i_{6,3*} = i_{7,2*} i_{6,1*}$ and $i_{6,1*}\pi_{10}(S^5) = 0$

by § 5.2 (*k*), whence $i_{6,3*}\pi_{10}(S^5) = 0$; and

(ii) $i_{6,3*}\pi_9(S^5)$ is non-trivial by (*c*) above.

Hence
$$\pi^5_{5,4} = Z_2,$$

generated by $\{i_{8,1}\,\mathrm{p}h_{7,10}\}$.

8.4. $k \equiv 3 \pmod 4$.

We have from § 2.3 (*b*) that

$$p_{k+3,1*}\{t_{k+4,4}\} = 2\{h_{k+2,k+2}\},$$

and so from § 7.2 (*b*) we have that $\{t_{k+4,4}\}$ generates an infinite summand of $\pi^2_{k,3} = Z_2 + Z_\infty$. Thus we have the following:

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+2*}} \pi^2_{k,4} \to \pi_{k+2}(S^{k+3}),$$

.e.
$$\to Z_\infty \to Z_2 + Z_\infty \to \pi^2_{k,4} \to 0,$$

by § 7.2 (*b*), where $i^{-1}_{k+2\#}(0)$ is generated by $\{t_{k+4,4}\}$. Using the above result we have that

$$\pi^2_{k,4} = Z_2,$$

generated by $i_{k+2,2\#}a$, where $p_{k+2,1\#}a = \{h_{k+1,k+2}\}$. Note that since $\Delta^{-1}_{\#}(0) = 0$, $p_{k+3\#}$ is trivial.

(*b*) When $p = 3$ and $k \geqslant 7$, (C) gives

$$\xrightarrow{p_{k+4\#}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_\#} \pi^3_{k,3} \xrightarrow{i_{k+3\#}} \pi^3_{k,4} \xrightarrow{p_{k+3\#}} \pi_{k+3}(S^{k+3}) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi^3_{k,4} \to 0,$$

by § 7.2 (*c*), and since $p_{k+3\#}$ is trivial by (*a*). Also

$$i^{-1}_{k+3\#}(0) = t_{k+4,4\#}\pi_{k+3}(S^{k+2}),$$

which is generated by $h^*_{k+2,k+2}\{t_{k+4,4}\}$. Any of the methods used previously in determining $i^{-1}_{r\#}(0)$ will be seen to break down in this case. All we can say at the moment is that $\pi^3_{k,4}$ is either Z_2 or $Z_2 + Z_2$. I shall prove that it is the latter.

To determine $h^*_{k+2,k+3}\{t_{k+4,4}\}$, we consider first the sequence associated with the fibring $V_{k+3,3}/S^k \to V_{k+3,2}$, and operate with $h^*_{k+2,k+3}$ on the section

$$\to \pi^2_{k,3} \xrightarrow{p_{k+3,2\#}} \pi^1_{k+1,2} \to,$$

giving the commutative diagram (3.1 *b*),

$$\begin{array}{ccc} \to \pi^3_{k,3} & \xrightarrow{p_{k+3,2\#}} & \pi^2_{k+1,2} \to \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} \\ \to \pi^2_{k,3} & \xrightarrow{p_{k+3,2\#}} & \pi^1_{k+1,2} \to. \end{array}$$

Then we have that

$$\begin{aligned} p_{k+3,2\#}h^*\{t_{k+4,4}\} &= h^* p_{k+3,2\#}\{t_{k+4,4}\} \\ &= h^*_{k+2,k+3}\{t_{k+4,3}\}, \quad \text{by § 2.3 } (b), \\ &= 0, \quad \text{by § 7.31 } (b). \end{aligned}$$

Hence

$$h^*\{t_{k+4,4}\} \in p^{-1}_{k+3,2\#}(0).$$

Together with the above, consider now the commutative diagram (2.1),

$$\begin{array}{ccc} \to \pi^2_{k,3} & \xrightarrow{p_{k+3,2\#}} & \pi^1_{k+1,2} \to \\ \uparrow{\scriptstyle i_{k+2,1\#}} & & \uparrow{\scriptstyle i_{k+2,1\#}} \\ \to \pi^2_{k,2} & \xrightarrow{p_{k+3,2\#}} & \pi_{k+2}(S^{k+1}) \to. \end{array}$$

We have from § 7.2 (*b*) that, if a' is the generator of finite order in $\pi^2_{k,3} = Z_2 + Z_\infty$, then $a' = i_{k+2,1*}a$, where a generates $\pi^2_{k,2} = Z_4$. Further

$$p_{k+3,2*}a = \{h_{k+1,k+2}\}$$

by § 5.2 (*c*). Thus

$$\begin{aligned} p_{k+3,2*}h^*a' &= h^*p_{k+3,2*}a' = h^*p_{k+3,2*}i_{k+2,1*}a \\ &= h^*i_{k+2,1*}p_{k+3,2*}a \\ &= i_{k+2,1*}h^*_{k+2,k+3}\{h_{k+1,k+2}\} \\ &= i_{k+2,1*}\{h_{k+1,k+3}\} \\ &\neq 0, \end{aligned}$$

since $V_{k+3,2}/S^{k+1} \to S^{k+2}$ admits a cross-section (5.1). Hence

$$h^*a' \neq 0; \text{ and further, } h^*a' \neq h^*\{t_{k+4,4}\}.$$

We now turn our attention to P^{k+2}_{k-1}. By Theorem 2.3 (*g*),

$$\psi_{k+4,4*}: \pi_{k+2}(P^{k+2}_{k-1}) \to \pi^2_{k,3}$$

is an isomorphism for $k > 2$, and

$$\psi_{k+4,4*}: \pi_{k+3}(P^{k+2}_{k-1}) \to \pi^3_{k,3},$$

is an isomorphism for $k > 3$. Further, it was shown in (B) of the Appendix to (**9**) that, for $k \equiv 3 \pmod 4$, P^{k+2}_{k-1} is of the same homotopy type as B^{k+2}_2, the space defined in Corollary 3.2 (C). Hence we have a homotopy equivalence $i: P^{k+2}_{k-1} \to B^{k+2}_2$ when $k \equiv 3 \pmod 4$. Thus, dropping unambiguous subscripts, we have

$$\begin{aligned} h^*i_*\psi_*^{-1}\pi^2_{k,3} &= h^*i_*\pi_{k+2}(P^{k+2}_{k-1}) \\ &= h^*\pi_{k+2}(B^{k+2}_2) \\ &= h^*\mathfrak{E}^{k-3}\pi_5(B^5_2), \quad \text{by Corollary 3.2 (C),} \\ &= \mathfrak{E}^{k-3}h^*_{5,6}\pi_5(B^5_2), \quad \text{by Lemma 3.1 } (b), \\ &\subset \mathfrak{E}^{k-3}\Omega, \end{aligned}$$

where Ω is the subgroup of $\pi_6(B^5_2)$ generated by the elements of order two. Now it will be shown in the Appendix that $\pi_6(B^5_2) = Z_4 + Z_2$, generated by $\bar{w}$ and $\bar{z}$ of orders 4 and 2 respectively, and further that

$\mathfrak{E}^2(2\bar{w}) = 0$. *Hence, since* $k \geqslant 7$, *there can be only one non-zero element in* $h^*i_*\psi_*^{-1}\pi^2_{k,3}$, *and that is* $\mathfrak{E}^{k-3}(\bar{z})$.

But $$h^*i_*\psi_*^{-1}\pi^2_{k,3} = i_*\psi_*^{-1}h^*\pi^2_{k,3},$$

and i_* and ψ_*^{-1} are both isomorphisms. Hence there can be only one non-zero element in $h^*\pi^2_{k,3}$. But we have from above that $h^*a' \neq 0$, and that $h^*a' \neq h^*\{t_{k+4,4}\}$.

Thus $$h^*_{k+2,k+3}\{t_{k+4,4}\} = 0.$$

From this it follows that $\pi^3_{k,4} \approx \pi^3_{k,3}$, since we saw that

$$\pi^3_{k,4} \approx \pi^3_{k,3}/(\text{the subgroup generated by } h^*_{k+2,k+3}\{t_{k+4,4}\}).$$

Thus $$\pi^3_{k,4} = Z_2 + Z_2,$$

generated by $\{i_{k+1,3}h_{k,k+3}\}$, and $i_{k+2,2*}a$, where

$$p_{k+2,1*}a = \{h_{k+1,k+3}\}.$$

Note that, since Δ_* is trivial, p_{k+4*} is onto.

(*c*) When $p = 3$ and $k = 3$, (C) gives

$$\xrightarrow{p_{7*}} \pi_7(S^6) \xrightarrow{\Delta_*} \pi^3_{3,3} \xrightarrow{i_{6*}} \pi^3_{3,4} \xrightarrow{p_{6*}} \pi_6(S^6) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^3_{3,4} \to 0,$$

by § 7.2 (*d*) and since p_{6*} is trivial by (*a*). Hence $\pi^3_{3,4}$ is at most Z_2. I now show that it is at least Z_2. Consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which, starting with the term $\pi^3_{3,4}$, is of the form

$$\to \pi^3_{3,4} \to \pi^2_{4,3} \to \pi_5(S^3) \to \pi^2_{3,4} \to \pi^1_{4,3} \to \pi_4(S^3) \to,$$

i.e. $$\to \pi^3_{3,4} \to Z_2 + Z_2 \to Z_2 \to Z_2 \to Z_2 + Z_2 \to Z_2 \to$$

by the results of (*a*) above and of § 7.31 (*a*) and (*b*). From the exactness of this sequence we have that $\pi^3_{3,4}$ is at least Z_2. Hence

$$\pi^3_{3,4} = Z_2,$$

and is generated by $i_{5,2*}p^{-1}_{5,1*}\{h_{4,6}\}$. Note that, since Δ_* is trivial, p_{7*} is onto.

Further, since $i^{-1}_{6*}(0) = t_{7,4*}\pi_6(S^5)$ and since it also is zero from above, we have that

$$h^*_{5,6}\{t_{7,4}\} = 0.$$

(*d*) When $p = 4$ and $k \geqslant 7$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}),$$

i.e. $$\to Z_2 \to Z_4 \to \pi^4_{k,4} \to Z_2 \to 0,$$

by § 7.2 (*e*) and since p_{k+4*} is trivial by (*b*). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{t_{k+4,4}\} = h^*_{k+3,k+4}h^*_{k+2,k+3}\{t_{k+4,4}\} = 0$$

by (*b*).

Hence $$\pi^4_{k,3} \textit{ is an extension of } Z_4 \textit{ by } Z_2.$$

Note that, since Δ_* is trivial, p_{k+5*} is onto.

To determine the extension consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{k+4}(S^k) & \xrightarrow{i_{k+1,3*}} & \pi^4_{k,4} & \xrightarrow{p_{k+4,3*}} & \pi^3_{k+1,3} & \xrightarrow{\Delta_*} & \pi_{k+3}(S^k) \to \\
\uparrow{\scriptstyle i_{k+1,0*}} & & \uparrow{\scriptstyle i_{k+3,1*}} & & \uparrow{\scriptstyle i_{k+3,1*}} & & \uparrow{\scriptstyle i_{k+1,0*}} \\
\to \pi_{k+4}(S^k) & \xrightarrow{i_{k+1,2*}} & \pi^4_{k,3} & \xrightarrow{p_{k+3,2*}} & \pi^3_{k+1,2} & \xrightarrow{\Delta_*} & \pi_{k+3}(S^k) \to,
\end{array}$$

in which the horizontal sequences are associated with the fibrings

$$V_{k+4,4}/S^k \to V_{k+4,3}, \qquad V_{k+3,3}/S^k \to V_{k+3,2},$$

and the $i_{k+1,0*}$ are isomorphisms. Also, by § 7.31 (*c*) and by (*d*) above, the $i_{k+1,3*}$ are isomorphisms into. Then, when we use the results of §§ 7.2 (*e*), 5.1, and 7.31 (*c*), the diagram becomes

$$\begin{array}{ccccccc}
\to 0 \to & \pi^4_{k,4} & \to & Z_{24}+Z_4 & \to & Z_{24} \to \\
\| & \uparrow & & \uparrow & & \| \\
\to 0 \to & Z_4 & \to & Z_{24}+Z_2 & \to & Z_{24} \to .
\end{array}$$

Let $a = \mathrm{p}_*\{h_{k+2,k+4}\}$, the generator of a Z_2 summand of $\pi^3_{k+1,2} = Z_{24}+Z_2$, and b the generator of a Z_4 summand of $\pi^3_{k+1,3} = Z_{24}+Z_4$ as defined in § 7.31 (*c*). Then $i_{k+3,1*}a = 2b$. Now from the exactness of the lower sequence, or alternatively from § 7.2 (*e*), it follows that

$$\Delta_* a = 12\{h_{k,k+3}\}.$$

Hence

$$\Delta_* 2b = \Delta_* i_{k+3,1*} a = i_{k+1,0*}\Delta_* a = i_{k+1,0*} 12\{h_{k,k+3}\}.$$

Thus

$$\Delta_* 2b = 12\{h_{k,k+3}\} \neq 0.$$

But, if $\pi^4_{k,4}$ were Z_4+Z_2, $2b$ would be in $p_{k+4,3*}\pi^4_{k,4}$ since $2b$ is of order two and there are just three elements of order two in each of $Z_{24}+Z_4$ and the isomorphic image of Z_4+Z_2. Hence $\Delta_* 2b$ would have to be zero, and we have just proved the contrary. Thus

$$\pi^4_{k,4} = Z_8 \quad \text{for } k \geqslant 7,$$

generated by a such that $p_{k+4,1*}a = \{h_{k+3,k+4}\}$.

(*e*) When $p = 4$ and $k = 3$, (C) gives

$$\xrightarrow{p_{8*}} \pi_8(S^6) \xrightarrow{\Delta_*} \pi^4_{3,3} \xrightarrow{i_{7*}} \pi^4_{3,4} \xrightarrow{p_{7*}} \pi_7(S^6) \to,$$

i.e.

$$\to Z_2 \to Z_\infty + Z_4 \to \pi^4_{3,4} \to Z_2 \to 0,$$

by § 7.2 (*f*) and since p_{7*} is onto by (*c*). Further,

$$i_{7*}^{-1}(0) = t_{7,4*}\pi_7(S^5),$$

which is generated by

$$h^*_{5,7}\{t_{7,4}\} = h^*_{6,7}\, h^*_{5,6}\{t_{7,4}\} = 0,$$

by (*c*). Hence

$$\pi^4_{3,4} \text{ is an extension of } Z_\infty + Z_4 \text{ by } Z_2.$$

Note that Δ_* is trivial, whence p_{8*} is onto.

We determine the extension as in (*d*) above by considering the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_7(S^3) & \xrightarrow{i_{4,3*}} & \pi^4_{3,4} & \xrightarrow{p_{7,3*}} & \pi^3_{4,3} & \xrightarrow{\Delta_*} & \pi_6(S^3) & \to \\
\uparrow{\scriptstyle i_{4,0*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{4,0*}} & \\
\to \pi_7(S^3) & \xrightarrow{i_{4,2*}} & \pi^4_{3,3} & \xrightarrow{p_{6,2*}} & \pi^3_{4,2} & \xrightarrow{\Delta_*} & \pi_6(S^3) & \to,
\end{array}$$

where the horizontal sequences are associated with the fibrings

$$V_{7,4}/S^3 \to V_{7,3}, \qquad V_{6,3}/S^3 \to V_{6,2},$$

and the $i_{4,0*}$ are isomorphisms. Also, by § 7.31 (*c*) and by (*e*) above, the $i_{6,1*}$ are isomorphisms into. We have further from § 7.2 (*f*) that $i_{4,2*}\pi_7(S^3) = 0$, whence it also follows that $i_{4,3*}\pi_7(S^3) = 0$. Then, by using the results of §§ 7.2 (*f*), 5.1, and 7.31 (*c*), the diagram becomes

$$\begin{array}{ccccccc}
0 \to & \pi^4_{3,4} & \to Z_\infty + Z_{12} + Z_4 & \to Z_{12} & \to \\
& \uparrow & \uparrow & \| & \\
0 \to Z_\infty + Z_4 & \to Z_\infty + Z_{12} + Z_2 & \to Z_{12} & \to. &
\end{array}$$

Let $a = \mathrm{p}_*\{h_{5,7}\}$, the generator of a Z_2 summand in $\pi^3_{4,2}$, and b the generator of a Z_4 summand of $\pi^3_{4,3}$ as defined in § 7.31 (*c*). Then

$$i_{6,1*}a = 2b.$$

Further, we have from § 7.2 (*f*) that $\Delta_* a = 6\{h_{3,6}\}$. Hence

$$\Delta_* 2b = \Delta_* i_{6,1*} a = i_{4,0*}\Delta_* a = i_{4,0*} 6\{h_{3,6}\}.$$

Thus

$$\Delta_* 2b = 6\{h_{3,6}\} \neq 0.$$

Now we had that $\pi^4_{3,4}$ is an extension of $Z_\infty + Z_4$ by Z_2; i.e. it is either $Z_\infty + Z_4 + Z_2$, $Z_\infty + Z_8$, or $Z_\infty + Z_4$. But, if $\pi^4_{3,4}$ were $Z_\infty + Z_4 + Z_2$, $2b$ would be in $p_{7,3*}\pi^4_{3,4}$ since $2b$ is of order two and there are just three elements of order two in each of $Z_\infty + Z_{12} + Z_4$ and the isomorphic image of $Z_\infty + Z_4 + Z_2$. Thus $\Delta_* 2b$ would have to be zero, and we have just proved the contrary. Thus $\pi^4_{3,4} \neq Z_\infty + Z_4 + Z_2$. Neither is it $Z_\infty + Z_8$ since it is impossible to map $Z_\infty + Z_8$ isomorphically into $Z_\infty + Z_{12} + Z_4$.

Hence

$$\pi^4_{3,4} = Z_4 + Z_\infty,$$

generated by $i_{6,1*}a$, where a is of order four and $p_{6,1*}a = \{h_{5,7}\}$, and b, where $p_{7,1*}b = \{h_{6,7}\}$, of infinite order.

(*f*) When $p = 5$ and $k \geqslant 7$, (C) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+3}) \xrightarrow{\Delta_*} \pi^5_{k,3} \xrightarrow{i_{k+5*}} \pi^5_{k,4} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \to,$$

i.e.
$$\to Z_{24} \to Z_{24} \to \pi^5_{k,4} \to Z_2 \to 0,$$

by § 7.2 (*g*) and since p_{k+5*} is onto by (*d*). Now we know from Theorem 4.2 (*b*) that $\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_2 + Z_2$ by § 7.31 (*d*). Hence

$$\pi^5_{k,4} = Z_2 + Z_2 \text{ when } k \geqslant 7,$$

and is generated by $i_{k+3,1*} p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$ and a such that

$$p_{k+4,1*} a = \{h_{k+3,k+5}\}.$$

Note that p_{k+6*} is thus onto the Z_2 subgroup of $\pi_{k+6}(S^{k+3})$.

(*g*) When $p = 5$ and $k = 3$, (C) gives

$$\xrightarrow{p_{9*}} \pi_9(S^6) \xrightarrow{\Delta_*} \pi^5_{3,3} \xrightarrow{i_{8*}} \pi^5_{3,4} \xrightarrow{p_{8*}} \pi_8(S^6) \to,$$

i.e.
$$\to Z_{24} \to Z_2 + Z_{24} \to \pi^5_{3,4} \to Z_2 \to 0,$$

by § 7.2 (*h*) and since p_{8*} is onto by (*e*). Further

$$i^{-1}_{8*}(0) = t_{7,4*}\pi_8(S^5),$$

which is generated by $h^*_{5,8}\{t_{7,4}\}$. But, by Lemma 3.1 (*b*), we have that

$$p_{6,1*} h^*_{5,8}\{t_{7,4}\} = h^*_{5,8} p_{6,1*}\{t_{7,4}\} = h^*_{5,8} 2\{h_{5,5}\}, \quad \text{by § 2.3 (b)},$$
$$= 2\{h_{5,8}\}.$$

Hence
$$h^*_{5,8}\{t_{7,4}\} = 2a + \lambda b \quad (\lambda = 0 \text{ or } 1),$$

where a generates a Z_{24} summand of $\pi^5_{3,3}$, and b a Z_2 summand.

To determine λ we consider the sequence associated with the fibring $V_{6,3}/S^3 \to V_{6,2}$, which is of the form

$$\to \pi^5_{3,3} \xrightarrow{p_{6,2*}} \pi^4_{4,2} \xrightarrow{\Delta_*} \pi_7(S^3) \to,$$

i.e.
$$\to Z_2 + Z_{24} \to Z_2 + Z_2 + Z_{24} \to Z_2 \to,$$

by § 5.1, together with the sequence of § 5.1

$$0 \to \pi_8(S^4) \to \pi^4_{4,2} \underset{\mathfrak{p}_*}{\overset{p_{6,1*}}{\rightleftarrows}} \pi_8(S^5) \to 0,$$

i.e.
$$0 \to Z_2 + Z_2 \to Z_2 + Z_2 + Z_{24} \longleftrightarrow Z_{24} \to 0.$$

Thus
$$p_{6,1*} p_{6,2*} a = p_{6,1*} a = \{h_{5,8}\}.$$

Hence
$$p_{6,2*} a = \mathfrak{p}_*\{h_{5,8}\} + w,$$

where w is of order two. So

$$p_{6,2*} 2a = 2\mathfrak{p}_*\{h_{5,8}\}.$$

Further, by § 2.4 (*b*), we have that

$$p_{6,2*}\{t_{7,4}\} = \{t_{7,3}\}.$$

Hence $$p_{6,2*}h^*_{5,8}\{t_{7,4}\} = h^*_{5,8}\{t_{7,3}\} = 2\mathfrak{p}_*\{h_{5,8}\},$$

by § 7.31 (*d*). Thus $$p_{6,2*}(2a+\lambda b) = p_{6,2*}2a.$$

But $p_{6,2*}: \pi^5_{3,3}$ is an isomorphism into by exactness. Thus $\lambda = 0$, and

$$h^*_{5,8}\{t_{7,4}\} = 2a.$$

Hence $i^{-1}_{8*}(0)$ is the subgroup generated by $2a$, whence

$$\pi^5_{3,4} \textit{ is an extension of } Z_2+Z_2 \textit{ by } Z_2.$$

Note that $\Delta_*\pi_9(S^6) = 2Z_{24}$, whence the image of p_{9*} is the Z_2 subgroup generated by $12\{h_{6,9}\}$.

To determine the extension, consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which is of the form:

$$\to \pi^5_{3,4} \xrightarrow{p_{7,3*}} \pi^4_{4,3} \xrightarrow{\Delta_*} \pi_7(S^3) \to,$$

i.e. $$\to \pi^5_{3,4} \to Z_2+Z_2+Z_2+Z_2 \to Z_2 \to$$

by § 7.31 (*d*). But $\pi^5_{3,4}$ has eight elements, whence by exactness $p_{7,3*}$ is an isomorphism into. Since $\pi^4_{4,3}$ has no elements of order greater than two, this implies that

$$\pi^5_{3,4} = Z_2+Z_2+Z_2,$$

and is generated by $i_{5,2*}p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$, $i_{6,1*}a$, where $p_{6,1*}a = \{h_{5,8}\}$, and b, where $p_{7,1*}b = \{h_{6,8}\}$.

8.5. $k = 1$.

Again we first calculate

$$\{t_{5,4}\} \in \pi^2_{1,3} = Z_\infty+Z_\infty, \text{ by § 7.1.}$$

To do this we define a projection $p': V_{4,3} \to V_{3,2}$ which maps $p^{-1}_{4,1}(s)$ homeomorphically onto $V_{3,2}$ for each $s \in S^3$ (which in R^4 has equation $x_0^2+x_1^2+x_2^2+x_3^2 = 1$). Let $v \in V_{4,3}$ be represented by

$$V = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \end{bmatrix}.$$

Now form $$V^* = \begin{bmatrix} -v_{14} & v_{13} & -v_{12} & v_{11} \\ -v_{13} & -v_{14} & v_{11} & v_{12} \\ v_{12} & -v_{11} & -v_{14} & v_{13} \end{bmatrix}.$$

Then $V_1 = V \times V^{*\prime}$ is easily seen to represent a point $v_1 \in V_{3,2}$. So we

define $p' : V_{4,3} \to V_{3,2}$ by $p'v = v_1$. Note that $p'i_{3,1} = 1 : V_{3,2} \to V_{3,2}$, and further, that, if the cross-section $\mathfrak{p} : S^3 \to V_{4,3}$ is chosen to be given by

$$\mathfrak{p}(x_0, x_1, x_2, x_3) = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ x_2 & x_3 & -x_0 & -x_1 \end{bmatrix},$$

then $p'\mathfrak{p}(S^3) = v_0$, the base point in $V_{3,2}$. Thus an element in $\pi_r(V_{4,3})$ is uniquely determined by its projection under p'_* and $p_{4,1*}$.

Now we have from § 2.2 that $t_{5,4}(x_0, x_1, x_2, x_3)$ is represented by

$$\begin{bmatrix} 2x_3x_0 & 2x_3x_1 & 2x_3x_2 & 2x_3^2-1 \\ 2x_2x_0 & 2x_2x_1 & 2x_2^2-1 & 2x_2x_3 \\ 2x_1x_0 & 2x_1^2-1 & 2x_1x_2 & 2x_1x_3 \end{bmatrix}.$$

Thus $p_{3,1}p't_{5,4}(x_0, x_1, x_2, x_3)$ is seen to be

$$\{2(x_2x_0+x_3x_1),\ 2(x_2x_1-x_3x_0),\ (x_3^2+x_2^2-x_1^2-x_0^2)\},$$

whence, by (7) 654, we see that $p_{3,1}p't_{5,4} : S^3 \to S^2$ is a map of Hopf invariant one. So

$$p_{3,1*}p'_*\{t_{5,4}\} = \{h_{2,3}\}.$$

But we have from § 2.3 (*b*) that $p_{4,1*}\{t_{5,4}\} = 2\{h_{3,3}\}$. Thus

$$\{t_{5,4}\} = i_{3,1*}p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathfrak{p}_*\{h_{3,3}\}.$$

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{4*}} \pi_4(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^2 \xrightarrow{i_{3*}} \pi_{1,4}^2 \to \pi_3(S^4),$$

i.e.

$$\to Z_\infty \to Z_\infty + Z_\infty \to \pi_{1,4}^2 \to 0,$$

by § 7.1. Also $i_{3*}^{-1}(0)$ is generated by $\{t_{5,4}\}$. Using the above result, we have that

$$\pi_{1,4}^2 = Z_\infty,$$

generated by $\{i_{4,1}\mathfrak{p}h_{3,3}\}$. Hence, by Theorem 4.2 (*a*), we also have that

$$\pi_{2,3}^1 = Z_\infty,$$

generated by $\{i_{4,1}\mathfrak{p}h_{3,3}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{4*} is trivial.

(*b*) When $p = 3$, (C) gives

$$\xrightarrow{p_{5*}} \pi_5(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^3 \xrightarrow{i_{4*}} \pi_{1,4}^3 \xrightarrow{p_{4*}} \pi_4(S^4) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,4}^3 \to 0,$$

by § 7.1 and since p_{4*} is trivial by (*a*). Also

$$i_{4*}^{-1}(0) = t_{5,4*}\pi_4(S^3),$$

which is generated by

$$h_{3,4}^*\{t_{5,4}\} = h_{3,4}^*(i_{3,1*}p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathfrak{p}_*\{h_{3,3}\}) = i_{3,1*}p_{3,1*}^{-1}\{h_{2,4}\} + 2\mathfrak{p}_*\{h_{3,4}\}$$
$$= i_{3,1*}p_{3,1*}^{-1}\{h_{2,4}\}.$$

Thus $i_{4*}^{-1}(0)$ is a Z_2 summand of $\pi_{1,3}^3$, whence

$$\pi_{1,4}^3 = Z_2,$$

generated by $\{i_{4,1}\, \mathrm{p} h_{3,4}\}$. Note that p_{5*} again is trivial. From Theorem 4.2 (a) we also have that

$$\pi_{2,3}^2 = Z_2,$$

generated by $\{i_{4,1}\, \mathrm{p} h_{3,4}\}$.

(c) When $p = 4$, (C) gives

$$\xrightarrow{p_{6*}} \pi_6(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^4 \xrightarrow{i_{5*}} \pi_{1,4}^4 \xrightarrow{p_{5*}} \pi_5(S^4) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,4}^4 \to 0,$$

by § 7.1 and since p_{5*} is trivial by (b). Also

$$i_{5*}^{-1}(0) = t_{5,4*}\, \pi_5(S^3),$$

which is generated by

$$\begin{aligned} h_{3,5}^*\{t_{5,4}\} &= h_{3,5}^*(i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathrm{p}_*\{h_{3,3}\}) \\ &= i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,5}\} + 2\mathrm{p}_*\{h_{3,5}\} = i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,5}\}. \end{aligned}$$

Thus $i_{5*}^{-1}(0)$ is a Z_2 summand of $\pi_{1,3}^4$, whence we have that

$$\pi_{1,4}^4 = Z_2,$$

generated by $\{i_{4,1}\, \mathrm{p} h_{3,5}\}$. Note that p_{6*} again is trivial. From Theorem 4.2 (a) we also have that

$$\pi_{2,3}^3 = Z_2,$$

generated by $\{i_{4,1}\, \mathrm{p} h_{3,5}\}$.

(d) When $p = 5$, (C) gives

$$\xrightarrow{p_{7*}} \pi_7(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^5 \xrightarrow{i_{6*}} \pi_{1,4}^5 \xrightarrow{p_{6*}} \pi_6(S^4) \to,$$

i.e.

$$\to Z_\infty + Z_{12} \to Z_{12} + Z_{12} \to \pi_{1,4}^5 \to 0,$$

by § 7.1 and since p_{6*} is trivial by (c).

Unfortunately in this case we cannot make use of Theorem 1.3 since $\pi_7(S^4) \neq \mathfrak{E}\pi_6(S^3)$. On the other hand, Theorem 1.4 gives us that $\Delta_*\, \mathfrak{E}\pi_6(S^3) = t_{5,4*}\, \pi_6(S^3)$ which is generated by

$$\begin{aligned} h_{3,6}^*\{t_{5,4}\} &= h_{3,6}^*(i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathrm{p}_*\{h_{3,3}\}) \\ &= h_{3,6}^*\, i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,3}\} + 2h_{3,6}^*\, \mathrm{p}_*\{h_{3,3}\} \\ &\qquad [h_{3,6}^* \text{ being a homomorphism since there is a} \\ &\qquad\quad \text{multiplication on } S^3\text{: see (4) Theorem IV}] \\ &= i_{3,1*}\, p_{3,1*}^{-1}\{h_{2,6}\} + 2\mathrm{p}_*\{h_{3,6}\}. \end{aligned}$$

Using the results of §§ 5.3 (b) and 7.1, we see that $\Delta_*\, \mathfrak{E}\pi_6(S^3)$ is the subgroup $\pi_{1,3}^5$ generated by $a + 2b$, where a and b are generators of complementary Z_{12} summands. The order of $\pi_{1,4}^5$ thus depends on $\Delta_*\, \bar{p}_*\{h_{7,7}\}$. I have not been able to evaluate this directly. However,

A. Borel and J.-P. Serre showed in (2) Proposition 19.3 that $\pi^5_{1,4} = 0$. From this it follows that $\Delta_* \bar{p}_*\{h_{7,7}\}$ is an element which with $a+2b$ generates $\pi^5_{1,3}$, and so is of order twelve. Hence the image of p_{7*} is the Z_∞ subgroup of $\pi^5_{1,3}$ generated by $12\bar{p}_*\{h_{7,7}\}$.

The method used by A. Borel and J.-P. Serre is as follows. It is well known that $V_{n,n-1}$ is homeomorphic to R_n, the rotation group of S^{n-1}, and that the universal covering group of R_5 is the second symplectic group Sp_2. Thus $\pi^5_{1,4} \approx \pi_6(Sp_2)$. Now Sp_2 fibres over S^7 with fibre Sp_1 [see (11) 25.5], and the sequence associated with this fibring is of the form

$$\to \pi_7(S^7) \xrightarrow{\Delta_*} \pi_6(S^3) \xrightarrow{i_{6*}} \pi_6(Sp_2) \to \pi_6(S^7)$$

since $Sp_1 = S^3$. Now $i^{-1}_{6*}(0)$ is generated by the characteristic map of the above fibring, and A. Borel and J.-P. Serre show in Proposition 19.1 of (2) that this characteristic map is a generator of $\pi_6(S^3)$. Thus $\pi_6(Sp_2) = 0$, whence it follows that

$$\pi^5_{1,4} = 0 \quad \text{and} \quad \pi^4_{2,3} = 0,$$

the latter by virtue of Theorem 4.2 (a).

(e) When $p = 6$, (C) gives

$$\xrightarrow{p_{8*}} \pi_8(S^4) \xrightarrow{\Delta_*} \pi^6_{1,3} \xrightarrow{i_{7*}} \pi^6_{1,4} \xrightarrow{p_{7*}} \pi_7(S^4) \to,$$

i.e.
$$\to Z_2 + Z_2 \to Z_2 + Z_2 \to \pi^6_{1,4} \to Z_\infty \to 0,$$

by § 7.1 and since the image of p_{7*} is a Z_∞ subgroup by (d). Again we cannot make use of Theorem 1.3. But, using the results of §§ 5.3 (b) and 7.1, we see that $\pi^6_{1,3}$ is generated by $i_{3,1*}p^{-1}_{3,1*}\{h_{2,7}\}$ and $\mathfrak{p}_*\{h_{3,7}\}$. Now

$$\begin{aligned} i_{7*}i_{3,1*}p^{-1}_{3,1*}\{h_{2,7}\} &= i_{7*}i_{3,1*}p^{-1}_{3,1*}h^*_{6,7}\{h_{2,6}\} \\ &= h^*_{6,7}i_{6*}i_{3,1*}p^{-1}_{3,1*}\{h_{2,6}\}, \quad \text{by Lemma 3.1 } (b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial by (d). Similarly

$$\begin{aligned} i_{7*}\mathfrak{p}_*\{h_{3,7}\} &= i_{7*}\mathfrak{p}_*h^*_{6,7}\{h_{3,6}\} \\ &= h^*_{6,7}i_{6*}\mathfrak{p}_*\{h_{3,6}\}, \quad \text{by Lemma 3.1 } (b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial. Thus $\pi^6_{1,4} = Z_\infty$,

generated by $p^{-1}_{5,1*}12\bar{p}_*\{h_{7,7}\}$. Again, from Theorem 4.2 (a) we have

$$\pi^5_{2,3} = Z_\infty,$$

generated by $p^{-1}_{5,1*}12\bar{p}_*\{h_{7,7}\}$. Note that, since i_{7*} is trivial, Δ_* is onto

$\pi^6_{1,3}$, whence it follows, since $\pi_8(S^4)$ is abstractly isomorphic to $\pi^6_{1,3}$, that $\Delta_*^{-1}(0) = 0$. Hence p_{8*} is trivial.

(*f*) When $p = 7$, (C) gives

$$\xrightarrow{p_{9*}} \pi_9(S^4) \xrightarrow{\Delta_*} \pi^7_{1,3} \xrightarrow{i_{8*}} \pi^7_{1,4} \xrightarrow{p_{8*}} \pi_8(S^4) \to,$$

i.e.

$$\to Z_2 + Z_2 \to Z_2 + Z_2 \to \pi^7_{1,4} \to 0,$$

by § 7.1 and since p_{8*} is trivial by (*e*). As above, instead of using Theorem 1.3, we note that, by §§ 5.3 (*b*) and 7.1, $\pi^7_{1,3}$ is generated by $i_{3,1*}\, p^{-1}_{3,1*}\{h_{2,8}\}$ and $\mathrm{p}_*\{h_{3,8}\}$. Again

$$\begin{aligned} i_{8*}\, i_{3,1*}\, p^{-1}_{3,1*}\{h_{2,8}\} &= i_{8*}\, i_{3,1*}\, p^{-1}_{3,1*}\, h^*_{6,8}\{h_{2,6}\} \\ &= h^*_{6,8}\, i_{6*}\, i_{3,1*}\, p^{-1}_{3,1*}\{h_{2,6}\}, \quad \text{by Lemma 3.1 } (b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial by (*d*). Similarly

$$\begin{aligned} i_{8*}\, \mathrm{p}_*\{h_{3,8}\} &= i_{8*}\, \mathrm{p}_*\, h^*_{6,8}\{h_{3,6}\} \\ &= h^*_{6,8}\, i_{6*}\, \mathrm{p}_*\{h_{3,6}\} \\ &= 0. \end{aligned}$$

Thus i_{8*} is trivial, whence it follows that

$$\pi^7_{1,4} = 0 \quad \text{and} \quad \pi^6_{2,3} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that, by an argument similar to that at the end of (*e*), we have that p_{9*} is trivial.

Appendix

(A) *The group* $\pi_6(P^4_2)$

We have from § 2 that $V_{5,2}$ fibres over S^4 with fibre S^3. It is thus a seven-dimensional space, and the cellular decomposition of J. H. C. Whitehead's in (**13**) shows that in fact, besides the P^4_2 embedded in it, $V_{5,2}$ contains just one other cell, an E^7. Thus the pair $(V_{5,2}, P^4_2)$ is of the type considered in (**14**), and its double sequence is of the form

$$\begin{array}{ccccccccc} \to \pi_7(E^7, S^6) & \to & \pi_6(S^6) & \to & \pi_6(E^7) & \to & \pi_6(E^7, S^6) & \to \\ \downarrow g_7 & & \downarrow h_6 & & & & \downarrow g_6 & \\ \to \pi_7(V_{5,2}, P^4_2) & \to & \pi_6(P^4_2) & \to & \pi_6(V_{5,2}) & \to & \pi_6(V_{5,2}, P^4_2) & \to. \end{array}$$

Now $\pi_7(E^7, S^6) = Z_\infty$ and $\pi_6(E^7, S^6) = 0$, and by Theorem I in (**14**) g_7 and g_6 are both isomorphisms. Also, by § 5.2 (*e*), $\pi_6(V_{5,2}) = Z_2$. Hence

$\pi_6(P^4_2)$ is an extension of Z_p by Z_2.

Consider next the commutative diagram

$$\begin{array}{ccccccccccc}
\to \pi_7(P_2^4, S^3) & \xrightarrow{\delta_*} & \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(P_2^4) & \xrightarrow{j_*} & \pi_6(P_2^4, S^3) & \xrightarrow{\delta_*} & \pi_5(S^3) \to \\
\downarrow{\scriptstyle p'_*\psi_{6,3*}} & & \downarrow{\scriptstyle \psi_{5,2*}} & & \downarrow{\scriptstyle \psi_{6,3*}} & & \downarrow{\scriptstyle p'_*\psi_{6,3*}} & & \downarrow{\scriptstyle \psi_{5,2*}} \\
\to \pi_7(S^4) & \xrightarrow{\Delta_*} & \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(V_{5,2}) & \xrightarrow{p_*} & \pi_6(S^4) & \xrightarrow{\Delta_*} & \pi_5(S^3) \to,
\end{array}$$

where the sequences are those of the pair (P_2^4, S^3) and the fibring $V_{5,2}/S^3 \to S^4$, and ψ and p'_* as defined in §§ 2.3 (*d*) and 1. From this follows first of all that $\delta_* \pi_6(P_2^4, S^3) = 0$, since $\psi_{5,2*}$ is an isomorphism and $\Delta_* \pi_6(S^4) = 0$ by § 5.2 (*c*).

Next we calculate $\pi_6(P_2^4, S^3)$. Let Y_2^{k+1} be the space consisting of an S^k to which one $(k+1)$-cell e^{k+1} has been attached by a map ϕ such that $\phi | \dot{E}^{k+1} \to S^k$ is of degree two. Then (Y_2^4, S^3) is of the same homotopy type as (P_2^4, S^3) [see (**9**) Appendix (A)]. Let S^{3*} be the closure of the space obtained from Y_2^4 by removing a 4-cell E^4 from the interior of the 4-cell of Y_2^4. Then (Y_2^4, S^{3*}) is of the same homotopy type as (Y_2^4, S^3). Consider thus the section of the upper sequence of the triad $(Y_2^4; E^4, S^{3*})$

$$\to \pi_6(E^4, \dot{E}^4) \xrightarrow{i_{5*}} \pi_6(Y_2^4, S^{3*}) \xrightarrow{j_{6*}} \pi_6(Y_2^4; E^4, S^{3*}) \xrightarrow{\delta_+} \pi_5(E^4, \dot{E}^4) \xrightarrow{i_{5*}}.$$

Since i_{r*} is homotopically equivalent to $g_r : \pi_r(E^4, \dot{E}^4) \to \pi_r(Y_2^4, S^3)$, the homomorphism induced by the attaching map ϕ, we have from Theorem 1.2 in (**6**) that both i_{5*} and i_{6*} are isomorphisms into, and moreover from Theorem 1.7 of the same paper that $i_{6*}\pi_6(E^4, \dot{E}^4)$ is a direct summand of $\pi_6(Y_2^4, S^{3*})$. Further, since $(S^{3*}, \dot{E}^4)$ is 2-connected and $(E^4, \dot{E}^4)$ is 3-connected, we have from Theorem 1 of (**1**) that

$$\pi_6(Y_2^4; E^4, S^{3*}) \approx \pi_3(S^{3*}, \dot{E}^4) \otimes \pi_4(E^4, \dot{E}^4).$$

But $\pi_4(E^4, \dot{E}^4) = Z_\infty$, and we have from the lower sequence of the above triad that $\pi_3(S^{3*}, \dot{E}^4) \approx \pi_3(Y_2^4, E^4)$, and from the homotopy sequence of the pair (Y_2^4, E^4) that $\pi_3(Y_2^4, E^4) \approx \pi_3(Y_2^4)$. Finally

$$\pi_3(Y_2^4) \approx \pi_3(P_2^4),$$

which by Theorem 2.3 (*g*) is isomorphic to $\pi_3(V_{5,2}) = Z_2$ by § 5.2 (*a*). Thus

$$\pi_6(Y_2^4; E^4, S^{3*}) \approx Z_2 \otimes Z_\infty = Z_2,$$

whence

$$\pi_6(P_2^4, S^3) \approx \pi_6(Y_2^4, S^3) = Z_2 + Z_2.$$

Finally consider the relative homotopy sequence of the triple $(V_{5,2}, P_2^4, S^3)$, which is of the form

$$\to \pi_7(P_2^4, S^3) \xrightarrow{p'_*\psi_{6,3*}} \pi_7(S^4) \xrightarrow{j_* p'^{-1}_*} \pi_7(V_{5,2}, P_2^4) \to \pi_6(P_2^4, S^3) \to,$$

i.e.

$$\to \pi_7(P_2^4, S^3) \to Z_\infty + Z_{12} \to Z_\infty \to Z_2 + Z_2 \to.$$

From this it follows by exactness that

$$p'_*\psi_{6,3*}\pi_7(P_2^4, S^3) = \mathfrak{E}\pi_6(S^3) \in \pi_7(S^4).$$

Hence, since $\psi_{5,2*}$ is an isomorphism, we have that

$$\delta_*\pi_7(P_2^4, S^3) = \psi_{5,2*}^{-1}\Delta_*\mathfrak{E}\pi_6(S^3) = 2\pi_6(S^3)$$

by § 5.2 (*e*). Thus $\quad i_*\pi_6(S^3) = Z_2.$

From the above it follows that $\pi_6(P_2^4)$ is an extension of Z_2 by Z_2+Z_2. But we saw that it was an extension of Z_p by Z_2. Hence

$$\pi_6(P_2^4) = Z_4+Z_2.$$

We need to know something about the suspension properties of one of the elements in $\pi_6(P_2^4) \approx \pi_6(Y_2^4)$. So consider the diagram

$$\begin{array}{ccccc} \to \pi_8(S^5) & \xrightarrow{i_{8*}} & \pi_8(Y_2^6) & \to \\ \uparrow{\scriptstyle \mathfrak{E}^2} & & \uparrow{\scriptstyle \mathfrak{E}^2} & \\ \to \pi_6(S^3) & \xrightarrow{i_{6*}} & \pi_6(Y_2^4) & \to \end{array}$$

and let w and z generate respectively the Z_4 and Z_2 summands of $\pi_6(Y_2^4)$. Then

$$2w = i_{6*}\{h_{3,6}\}.$$

Hence $\quad \mathfrak{E}^2 2w = \mathfrak{E}^2 i_{6*}\{h_{3,6}\} = i_{8*}\mathfrak{E}^2\{h_{3,6}\} = i_{8*}2\{h_{5,8}\} = 0,$

by (**14**) 4.5, since the attaching map is of degree two.

Thus $\quad \mathfrak{E}^2 2w = 0.$

(B) *The group* $\pi_6(B_2^5)$

Let B_2^{k+2} be the space consisting of Y_2^{k+1} to which one $(k+2)$-cell has been attached by a map ϕ such that $\phi|\dot{E}^{k+2}: \dot{E}^{k+2} \to S^k \subset Y_2^{k+1}$ and is essential. Then it was shown in the appendix (B) of (**9**) that (B_2^{k+2}, Y_2^{k+1}) is of the same homotopy type as $(P_{k-1}^{k+2}, P_{k-1}^{k+1})$ when $k \equiv 3 \pmod 4$. Let

$$q: (B_2^5, Y_2^4) \to (P_2^5, P_2^4)$$

be a homotopy equivalence. Then we have the commutative diagram

$$\begin{array}{ccccccc} \to \pi_6(Y_2^4) & \xrightarrow{i'_{6*}} & \pi_6(B_2^5) & \xrightarrow{j'_{6*}} & \pi_6(B_2^5, Y_2^4) & \to \\ \downarrow{\scriptstyle \psi_{6,3*}q_*} & & \downarrow{\scriptstyle \psi_{7,4*}q_*} & & \downarrow{\scriptstyle p'_*\psi_{7,4*}q_*} & \\ \to \pi_6(V_{5,2}) & \xrightarrow{i_{6*}} & \pi_6(V_{6,3}) & \xrightarrow{p_{6*}} & \pi_6(S^5) & \to. \end{array}$$

We have that

(*a*) $\psi_{7,4*}: \pi_6(P_2^5, P_2^4) \to \pi_6(V_{6,3}, V_{5,2})$ is an isomorphism by Lemma 2.3 (*f*), and so are p'_* and q_*. But p_{6*} is trivial by § 7.2 (*b*).

Thus $\quad j'_{6*}$ *is trivial.*

(*b*) By Theorem 1 in (3), $i''^{-1}_{6*}(0)$ is the union of $\phi_*\pi_6(S^4)$ and $[\phi_*\pi_4(S^4), \pi_3(Y^4_2)]$, where ϕ is the attaching map of the 5-cell in B^5_2. Thus

$$\phi_*\pi_6(S^4) = \bar{\imath}_* h_{3,4*}\pi_6(S^4) = \bar{\imath}_* 6\pi_6(S^3) = 0$$

by (A) above. Also $\pi_3(Y^4_2) = \bar{\imath}_*\pi_3(S^3)$

since $\pi_3(Y^4_2, S^3) \approx \pi_2(S^3) = 0$. Thus

$$\begin{aligned}[\phi_*\pi_4(S^4), \pi_3(Y^4_2)] &= [\bar{\imath}_* h_{3,4*}\pi_4(S^4), \bar{\imath}_*\pi_3(S^3)] \\ &= \bar{\imath}_*[\pi_4(S^3), \pi_3(S^3)] \\ &= 0\end{aligned}$$

since all Whitehead products vanish on S^3.

Thus $$i''^{-1}_{6*}(0) = 0.$$

Now let $\bar{w} = i'_{6*}w$, of order four, and $\bar{z} = i'_{6*}z$, of order two. Then

$$2\mathfrak{E}^2\bar{w} = \mathfrak{E}^2 i''_{6*}\, 2w = i''_{8*}\,\mathfrak{E}^2\, 2w = 0$$

by (A) above.

Hence $\pi_6(B^2_5) = Z_4 + Z_2$, *generated by* $\bar{w}$ *and* $\bar{z}$, *where* $\mathfrak{E}^2\, 2\bar{w} = 0$.

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From this it follows by exactness that

$$p'_*\psi_{6,3*}\pi_7(P_2^4, S^3) = \mathfrak{E}\pi_6(S^3) \in \pi_7(S^4).$$

Hence, since $\psi_{5,2*}$ is an isomorphism, we have that

$$\delta_*\pi_7(P_2^4, S^3) = \psi_{5,2*}^{-1}\Delta_*\mathfrak{E}\pi_6(S^3) = 2\pi_6(S^3)$$

by § 5.2 (*e*). Thus $\qquad i_*\pi_6(S^3) = Z_2.$

From the above it follows that $\pi_6(P_2^4)$ is an extension of Z_2 by Z_2+Z_2. But we saw that it was an extension of Z_p by Z_2. Hence

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and let w and z generate respectively the Z_4 and Z_2 summands of $\pi_6(Y_2^4)$. Then

$$2w = i_{6*}\{h_{3,6}\}.$$

Hence $\qquad \mathfrak{E}^2 2w = \mathfrak{E}^2 i_{6*}\{h_{3,6}\} = i_{8*}\mathfrak{E}^2\{h_{3,6}\} = i_{8*}2\{h_{5,8}\} = 0,$

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$$q: (B_2^5, Y_2^4) \to (P_2^5, P_2^4)$$

be a homotopy equivalence. Then we have the commutative diagram

$$\begin{array}{ccccc} \to \pi_6(Y_2^4) & \xrightarrow{i'_{6*}} & \pi_6(B_2^5) & \xrightarrow{j'_{6*}} & \pi_6(B_2^5, Y_2^4) \to \\ \downarrow\psi_{6,3*}q_* & & \downarrow\psi_{7,4*}q_* & & \downarrow p'_*\psi_{7,4*}q_* \\ \to \pi_6(V_{5,2}) & \xrightarrow{i_{6*}} & \pi_6(V_{6,3}) & \xrightarrow{p_{6*}} & \pi_6(S^5) \to. \end{array}$$

We have that

(*a*) $\psi_{7,4*}: \pi_6(P_2^5, P_2^4) \to \pi_6(V_{6,3}, V_{5,2})$ is an isomorphism by Lemma 2.3 (*f*), and so are p'_* and q_*. But p_{6*} is trivial by § 7.2 (*b*).

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$$\phi_*\pi_6(S^4) = \bar{\imath}_* h_{3,4*}\pi_6(S^4) = \bar{\imath}_* 6\pi_6(S^3) = 0$$

by (A) above. Also $\pi_3(Y^4_2) = \bar{\imath}_*\pi_3(S^3)$

since $\pi_3(Y^4_2, S^3) \approx \pi_2(S^3) = 0$. Thus

$$\begin{aligned}[\phi_*\pi_4(S^4), \pi_3(Y^4_2)] &= [\bar{\imath}_* h_{3,4*}\pi_4(S^4), \bar{\imath}_*\pi_3(S^3)] \\ &= \bar{\imath}_*[\pi_4(S^3), \pi_3(S^3)] \\ &= 0\end{aligned}$$

since all Whitehead products vanish on S^3.

Thus $$i''^{-1}_{6*}(0) = 0.$$

Now let $\bar{w} = i''_{6*}w$, of order four, and $\bar{z} = i''_{6*}z$, of order two. Then

$$2\mathfrak{E}^2\bar{w} = \mathfrak{E}^2 i''_{6*}2w = i''_{8*}\mathfrak{E}^2 2w = 0$$

by (A) above.

Hence $\pi_6(B^5_2) = Z_4 + Z_2$, *generated by* $\bar{w}$ *and* $\bar{z}$, *where* $\mathfrak{E}^2 2\bar{w} = 0$.

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ISOMORPHIC RINGS OF INFINITE MATRICES

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1. Introduction

LET $\Sigma(\alpha)$ be the set of all infinite matrices which map a sequence space α into itself, all the series which arise in the transformations being absolutely convergent. Then $\Sigma(\alpha)$ is a ring if α contains ϕ and is normal, or, in particular, if α is perfect [(**3**) 313]. When $\alpha \supseteq \phi$ and $\Sigma(\alpha)$ is a ring, then [(**3**) 312]

$$\Sigma(\alpha) \subseteq \Sigma(\alpha^{**}).$$

There are spaces α which contain ϕ and are not normal for which $\Sigma(\alpha)$ is a ring, and there are spaces α such that $\Sigma(\alpha)$ is not a ring [(**3**) 307, 313].

It is known that, if α and β are normal and contain ϕ and are homeomorphic in the Köthe–Toeplitz sense, then the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre [(**3**) 323]. In this case the homeomorphic mapping of α on β is induced by a matrix A, and the isomorphism between the rings can be written $\Sigma(\alpha) = A^{-1}\Sigma(\beta)A$. In this paper the converse of this result is established. It will be proved that, if α and β contain ϕ, and if $\Sigma(\alpha)$ and $\Sigma(\beta)$ are rings which are isomorphic over the centre, then (i) the spaces α and β are homeomorphic, (ii) the spaces α^* and β^* are homeomorphic, (iii) the isomorphism between the rings can be written

$$\Sigma(\alpha) = A^{-1}\Sigma(\beta)A,$$

where A is an infinite matrix whose transpose maps β^* on α^*. It will follow that, if α and β contain ϕ and are normal, then the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre if and only if the spaces α and β are homeomorphic.

The essential step in the proof of these results follows as a particular case of a theorem of Jacobson's which establishes that isomorphism between certain rings of transformations on two vector spaces implies the existence of a one-to-one semi-linear transformation of one vector space on the other [(**4**) 266].

In order to ensure that some of the transformations which arise are induced by matrices of the type $A = (a_{n,k})$ $(n, k = 1, 2, \ldots)$, some

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results on the transformations of sequence spaces are established in § 3. These are generalizations of known results [(3) 287, 294].

2. Notation and definitions

In accordance with the notation conventional in matrix theory, I shall define the product AB of two transformations A and B in a sequence space α by the relation

$$(AB)x = A(Bx) \quad (x \in \alpha).$$

A transformation is said to be *linear* when it preserves the algebraic operations in the vector space.

The *scalar product* $\sum_{i=1}^{\infty} x_i y_i$ of two vectors, one $x = \{x_i\}$ belonging to a sequence space α and the other $y = \{y_i\}$ belonging to the Köthe–Toeplitz dual space α^* of α, will be denoted by (x, y).

Two rings of matrices are said to be *isomorphic over the centre* if (i) the rings are isomorphic, (ii) the isomorphic mapping leaves scalar matrices invariant [(3) 321, 322].

3. Transformations of sequence spaces

We shall require the following lemma:

LEMMA. *If* (i) $\alpha \supseteq \phi, \beta \supseteq \phi$, (ii) $\phi \subseteq \alpha' \subseteq \alpha^*, \phi \subseteq \beta' \subseteq \beta^*$, (iii) L *is a linear transformation of* α *into* β *such that* $\alpha\alpha'-\lim x^{(n)} = x \in \alpha$ *implies* $\beta\beta'-\lim Lx^{(n)} = Lx$, *then there exists a unique matrix* A *such that* $Ax = Lx$ *for every* x *in* α.

Let $e^{(n)}$ denote the unit vector whose coordinates are given by $e_n^{(n)} = 1$, $e_k^{(n)} = 0$ if $k \neq n$. Let $Le^{(k)} = a^{(k)} = \{a_{1,k}, a_{2,k}, \ldots\}$, and let A be the matrix whose kth column vector is $a^{(k)}$ $(k = 1, 2, \ldots)$.

Suppose that $x = \{x_i\} \in \alpha$, and let $x^{(n)} = \sum_{i=1}^{n} x_i e^{(i)}$. Then [(2) 289]

$$\alpha\alpha'-\lim x^{(n)} = x,$$

and hence

$$\beta\beta'-\lim Lx^{(n)} = Lx.$$

We have

$$Lx^{(n)} = \sum_{i=1}^{n} x_i a^{(i)}, \qquad (Lx^{(n)})_p = \sum_{i=1}^{n} x_i a_{p,i} = (Ax^{(n)})_p.$$

Hence $Lx^{(n)} = Ax^{(n)}$, and the result follows from coordinate convergence.

THEOREM 1. *If* α *and* β *contain* ϕ *and are homeomorphic, there exists a matrix* A *such that* $\beta = A\alpha$, *and* A *has a unique two-sided reciprocal* A^{-1}, *which maps* β *on* α.

There is a one-to-one linear transformation L such that $L\alpha = \beta$, and L has the property that $\alpha\text{—}\lim x^{(n)} = x \in \alpha$ implies $\beta\text{—}\lim Lx^{(n)} = Lx$. The transformation L has an inverse L^{-1} which maps β on α. It follows from the lemma that there is a matrix A such that $Ax = Lx$ $(x \in \alpha)$, and a matrix B such that $By = L^{-1}y$ $(y \in \beta)$. If $x \in \alpha$, we have $B(Ax) = L^{-1}(Lx) = x$. Put $x = e^{(i)}$ $(i = 1, 2, \ldots)$ and we have $BA = I$, the unit matrix. Similarly $AB = I$. We observe that, in this case, the absolute convergence of the series which arise in the products BA, AB may not follow, as it does in the case when α and β are normal [(**3**) 294].

4. Isomorphic rings

THEOREM 2. *If* (i) $\alpha \supseteq \phi$, $\beta \supseteq \phi$, (ii) $\phi \subseteq \alpha' \subseteq \alpha^*$, $\phi \subseteq \beta' \subseteq \beta^*$, (iii) g *is an isomorphism over the centre of the ring* $\Sigma_{\alpha'}(\alpha)$ *on the ring* $\Sigma_{\beta'}(\beta)$, *then there exists a matrix* A *with a two-sided reciprocal* A^{-1} *such that*

$$\text{(i)} \quad A\alpha = \beta, \quad A'\beta' = \alpha',$$

$$\text{(ii)} \quad A^{-1}\beta = \alpha, \quad (A^{-1})'\alpha' = \beta',$$

$$\text{(iii)} \quad g(X) = AXA^{-1}, \quad \textit{for all } X \in \Sigma_{\alpha'}(\alpha).$$

The ring of all transformations induced in α by matrices in $\Sigma_{\alpha'}(\alpha)$ is the ring $\mathscr{L}(\alpha'|\alpha)$ of all linear transformations in α which have transposes in the dual space α' [(**1**) 503], and the ring of transformations induced in β by matrices in $\Sigma_{\beta'}(\beta)$ is the ring $\mathscr{L}(\beta'|\beta)$. The transformation induced in α by a matrix $X \in \Sigma_{\alpha'}(\alpha)$ will be denoted by T_X.

We may regard g as an isomorphism of the ring $\mathscr{L}(\alpha'|\alpha)$ on the ring $\mathscr{L}(\beta'|\beta)$. It follows that there is a one-to-one semi-linear transformation U of α on β, whose transpose U' maps β' on α', such that [(**4**) 266]

$$g(T_X) = UT_X U^{-1}.$$

To the transformation U corresponds an automorphism u in the complex field $\mathscr{C}$ such that $U(ax) = a^u Ux$, for every α in $\mathscr{C}$ and x in α [(**4**) 236]. If $v \in \beta$, $a \in \mathscr{C}$, we have

$$av = (aI)v = g(aI)v = (UT_{aI}U^{-1})v = a^u v.$$

Thus $a^u = a$ for every a in $\mathscr{C}$, and it follows that U is linear. Suppose that

$$\alpha\alpha'\text{—}\lim x^{(n)} = x \in \alpha.$$

Then, if $y \in \beta'$, we have

$$\lim_{n\to\infty} (Ux^{(n)}, y) = \lim_{n\to\infty} (x^{(n)}, U'y) = (x, U'y) = (Ux, y). \tag{1}$$

Hence $\beta\beta'\text{—}\lim Ux^{(n)} = Ux$. It follows from the lemma that there is a matrix $A = (a_{n,k})$ such that $Ax = Ux$ for all x in α. The transforma-

tion U^{-1} is a linear transformation of β on α, and it follows in the same way that there is a matrix $B = (b_{n,k})$ such that $U^{-1}y = By$ for all y in β. Similarly there are matrices $E = (e_{n,k})$ and $F = (f_{n,k})$ such that $U'v = Ev$ and $(U')^{-1}z = Fz$, for all v in β' and z in α'.

Since A maps α on β, it follows that the column vectors of A are in β. Similarly the column vectors of B are in α. If $y \in \beta$, then

$$A(By) = U(U^{-1}y) = y.$$

Taking $y = e^{(i)}$ $(i = 1, 2, \ldots)$, we obtain $AB = I$. Similarly $BA = I$. Thus $B = A^{-1}$.

If $x \in \alpha$ and $v \in \beta'$, we have $(Ax, v) = (Ux, v) = (x, U'v) = (x, Ev)$. Taking $x = e^{(i)}$, $v = e^{(j)}$ $(i, j = 1, 2, \ldots)$, we obtain $E = A'$. Similarly $F = B'$. This proves (i) and (ii).

If $y \in \beta$ and $X \in \Sigma_{\alpha'}(\alpha)$, we have

$$g(X)y = g(T_X)y = (UT_X U^{-1})y = U\{T_X(U^{-1}y)\} = A\{X(A^{-1}y)\}. \quad (2)$$

I shall prove that the product AXA^{-1} is associative. The column vectors of A' are in α', and, since X' maps α' into α', it follows that the column vectors of $X'A'$ are in α'. The row vectors of $(A^{-1})' = B'$ are column vectors of B and belong to α. It follows that $(A^{-1})'(X'A')$, and hence $(AX)A^{-1}$, exists. Similarly the product $A(XA^{-1})$ exists. We have

$$\{AX.A^{-1}\}_{i,j} = (A^{-1}e^{(j)}, (X'A')e^{(i)}) = (XA^{-1}e^{(j)}, A'e^{(i)}) = \{A.XA^{-1}\}_{i,j}. \quad (3)$$

Thus AXA^{-1} is associative, and, from (3), we obtain

$$\begin{aligned}(A\{X(A^{-1}y)\}, e^{(i)}) = (A^{-1}y, (X'A')e^{(i)}) &= (y, \{(A^{-1})'.X'A'\}e^{(i)}) \\ &= ((AXA^{-1})y, e^{(i)}).\end{aligned}$$

It follows, from (2), that $g(X)y = (AXA^{-1})y$ for all y in β. Taking $y = e^{(i)}$ $(i = 1, 2, \ldots)$ we obtain $g(X) = AXA^{-1}$. This proves the theorem.

THEOREM 3. *If* (i) $\alpha \supseteq \phi$, $\beta \supseteq \phi$, (ii) $\Sigma(\alpha)$ *and* $\Sigma(\beta)$ *are rings which are isomorphic over the centre, then the spaces* α *and* β *are homeomorphic, and the spaces* α^* *and* β^* *are homeomorphic.*

We have [(3) 309, 312]

$$\Sigma(\alpha) = \Sigma_{\alpha^*}(\alpha), \qquad \Sigma(\beta) = \Sigma_{\beta^*}(\beta).$$

We use Theorem 2 with $\alpha' = \alpha^*$ and $\beta' = \beta^*$. It follows that there is a linear transformation U and that $U\alpha = \beta$, $U^{-1}\beta = \alpha$, and that, by equations (1) of Theorem 2, U is continuous under projective

convergence [(3) 274]. Similarly U^{-1} is continuous under projective convergence. Hence α and β are homeomorphic.

Theorem 2 also shows that there exists a matrix A such that $A'\beta^* = \alpha^*$ and $(A^{-1})'\alpha^* = \beta^*$. Hence, if u and v are in β^* and $A'u = A'v$, then

$$(A^{-1})'(A'u) = (A^{-1})'(A'v)$$

and $u = v$. The change in the order of summation implied here is justified [(3) 284]. Thus the mapping of β^* on α^* is 1–1. The transformations are continuous [(3) 284]. This proves the theorem.

We observe that the isomorphism between the rings can be written

$$\Sigma(\alpha) = A^{-1}\Sigma(\beta)A.$$

THEOREM 4. *If α and β contain ϕ and are normal, the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre if and only if α and β are homeomorphic.*

It is known that, if α and β are homeomorphic, then the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre [(3) 323]. Theorem 3 establishes the converse.

THEOREM 5. *If* (i) *α and β contain ϕ,* (ii) *$\Sigma(\alpha)$ and $\Sigma(\beta)$ are rings which are isomorphic over the centre, then the rings $\Sigma(\alpha^{**})$ and $\Sigma(\beta^{**})$ are isomorphic over the centre.*

We observe [(3) 312] that $\Sigma(\alpha)$ is a subring of $\Sigma(\alpha^{**})$, and that $\Sigma(\beta)$ is a subring of $\Sigma(\beta^{**})$. The spaces α^* and β^* are homeomorphic, by Theorem 2, and hence α^{**} and β^{**} are homeomorphic [(3) 295]. The isomorphism between the rings $\Sigma(\alpha^{**})$ and $\Sigma(\beta^{**})$ follows [(3) 323].

COROLLARY. *If* (i) *α and β contain ϕ and are normal,* (ii) *the rings $\Sigma(\alpha)$ and $\Sigma(\beta)$ are isomorphic over the centre, then the rings $\Sigma(\alpha^{**})$ and $\Sigma(\beta^{**})$ are isomorphic over the centre.*

This is an important special case of Theorem 5 and follows immediately.

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ON OTTER'S FORMULA FOR ENUMERATING TREES

By L. E. CLARKE (*University College of Ghana*)

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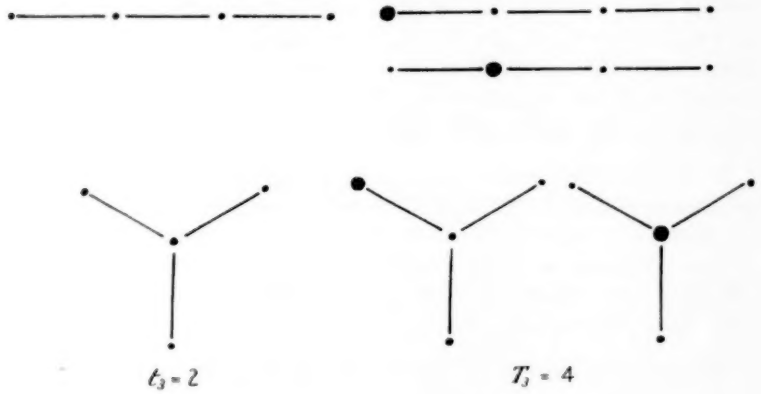
1. A TREE is a finite, connected graph with no closed paths; a tree has therefore one more vertex than it has lines. The number of lines of the tree will be denoted by n. The tree is *free* if its vertices are regarded as indistinguishable, and *rooted* if one of its vertices (the *root*) is regarded as distinguishable from the others (which are indistinguishable among themselves). Let t_n, T_n denote respectively the number of free and



rooted trees having n lines ($n = 1, 2, \ldots$). Write

$$t(x) = \sum_{n=0}^{\infty} t_n x^n, \qquad T(x) = \sum_{n=0}^{\infty} T_n x^n$$

(where $t_0 = T_0 = 1$).

Then it has been shown [see (**3**), (**1**)] that

$$t(x) = T(x) - \tfrac{1}{2}x\{T^2(x) - T(x^2)\}. \tag{1}$$

The object of this paper is to give a proof of (1) by an elementary argument which makes use of the concept of the 'centre of mass' of a tree. This proof of (1) seems to be somewhat simpler than either of the two mentioned above.

I shall show that, if $n = 2m$,

$$t_n = T_n - (T_0 T_{n-1} + T_1 T_{n-2} + \ldots + T_{m-1} T_m), \tag{2}$$

and, if $n = 2m+1$,

$$t_n = T_n-(T_0T_{n-1}+T_1T_{n-2}+\ldots+T_{m-1}T_{m+1}+T_m^2)+\tfrac{1}{2}(T_m^2+T_m). \quad (3)$$

These two sets of equations are together equivalent to (1).

2. If P is a vertex of the tree and PQ a line of the tree ending in P, then the *branch* (P, PQ) of the tree consists of P together with all vertices which can be reached from P by paths going through Q and all lines of such paths. A branch of a tree is thus a sub-tree of that tree. If $PQ_1,\ldots, PQ_k$ are the lines of the tree which end in P, and if the branch (P, PQ_i) contains h_i lines $(i = 1,\ldots,k)$, the *weight* of the vertex P is $\max(h_1,\ldots,h_k)$. The *centre of mass* of the tree is the set of those vertices which are of minimum weight.

It can be shown [(**2**) 70–75] that either there is a single point M of weight not exceeding $\frac{1}{2}n$, or there are two adjacent points M_1, M_2 of weight $\frac{1}{2}(n+1)$. In the first case the centre of mass consists of the single point M. In the second case (which can occur only if n is odd) the centre of mass consists of the pair of points M_1, M_2, and the branches (M_1, M_1M_2), (M_2, M_2M_1) each contain $\frac{1}{2}(n+1)$ lines.

3. The free trees with n lines having a single point M for the centre of mass can be obtained from the rooted trees with n lines having M as the root by rejecting those in which one of the branches ending in M has more than $\frac{1}{2}n$ lines. Clearly there can be only one such branch, (M, MN) say. The rooted trees in which (M, MN) has r $(> \frac{1}{2}n)$ lines can be obtained by taking the line MN and attaching

(i) a rooted tree with $r-1$ lines and root N to the vertex N,

(ii) a rooted tree with $n-r$ lines and root M to the vertex M.

This can be done in $T_{r-1}T_{n-r}$ ways. Thus the number of free trees whose centre of mass consists of a single point is

$$T_n-(T_0T_{n-1}+T_1T_{n-2}+\ldots+T_{m-1}T_m) \quad (n = 2m),$$

$$T_n-(T_0T_{n-1}+T_1T_{n-2}+\ldots+T_{m-1}T_{m+1}+T_m^2) \quad (n = 2m+1). \quad (4)$$

(In each case r takes the values $m+1,\ldots, n$.) This completes the proof of (2).

The free trees with n lines (where now $n = 2m+1$) having the two points M_1, M_2 for the centre of mass can be obtained by attaching rooted trees with m lines and roots M_1, M_2 to the vertices M_1, M_2 respectively of the line M_1M_2. This can be done in $\frac{1}{2}T_m(T_m-1)$ or T_m ways according as the rooted trees attached to M_1, M_2 are different or

identical. (In the resulting free tree M_1, M_2 will be regarded as indistinguishable; hence the factor $\frac{1}{2}$ in the first of the above terms.) Thus the number of free trees with n $(= 2m+1)$ lines whose centre of mass consists of a pair of points is

$$\tfrac{1}{2}T_m(T_m+1). \tag{5}$$

Equation (3) now follows from (4) and (5).

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A COUNTER-EXAMPLE IN THE GEOMETRY OF NUMBERS

By A. C. WOODS (*Tulane University, New Orleans*)

[Received 1 November 1957]

LET K be a convex body in R_n, symmetric in the hyperplane $x_1 = 0$. Further let K^* be the convex body in R_{n+1} obtained when K is rotated about $x_1 = 0$. Dr. Whitworth has pointed out to me that, in all the cases where the critical determinant of such a body K^* is known, there exists at least one critical lattice of K^* containing a critical lattice of K and has questioned whether this is always true. I give here an example to show that it is in general false. The argument is based on the three-dimensional case of a theorem of Swinnerton-Dyer's (**1**), which asserts that any critical lattice of a convex body in R_n has at least $n(n+1)$ of its points on the boundary of the body.

Let a, b be a pair of positive numbers such that $\frac{1}{3}b < a < b$. Form the hexagon K in R_2 with vertices $(a, 1)$, $(b, 0)$, $(a, -1)$, $(-a, -1)$, $(-b, 0)$, $(-a, 1)$. K is symmetric in the line $x_1 = 0$. Hence there is a K^* associated with K. I assert that no critical lattice of K^* contains a critical lattice of K. For assume that this assertion is false. It is well known that there is just one critical lattice Λ of K, and Λ is generated by the points $(0, 1)$, $(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2})$. By hypothesis there exists a critical lattice Λ^* of K^* containing Λ. All the points of Λ^* lie in hyperplanes of the form $x_3 = nc$ $(n = 0, \pm 1, \pm 2, \ldots)$, where c is some positive number. Evidently $d(\Lambda^*) = cd(\Lambda) = \frac{1}{2}c(a+b)$. All the points of Λ lie on lines of the form

$$x_1 = \tfrac{1}{2}n(a+b), \qquad x_3 = 0 \quad (n = 0, \pm 1, \pm 2, \ldots),$$

and on each of these lines the points of Λ are spaced at unit distance apart. Since $2a > \frac{1}{2}(a+b)$, it follows firstly that there is at least one point of Λ^* of the form (x_1', x_2', c) with the property that $|x_1'| < a$; and secondly that the points of the form $(x_1', x_2'+n, c)$ $(n = 0, \pm 1, \pm 2, \ldots)$ also have this property.

The part of K^* which lies between the hyperplanes $|x_1| = a$ is a right circular cylinder of unit radius. I assert that $c \geqslant \sqrt{\frac{3}{4}}$. For assume that $c < \sqrt{\frac{3}{4}}$. The line $(x_1', x_2'+t, c)$ intersects K^* in the points $\{x_1', \sqrt{(1-c^2)}, c\}$ which are at a distance $2\sqrt{(1-c^2)}$ apart. But $2\sqrt{(1-c^2)} > 1$, which implies that at least one of the points $(x_1', x_2'+n, c)$ of Λ^* is in the interior

of K^*. Since Λ^* is a critical lattice of K^*, this is impossible. Therefore $c \geqslant \sqrt{\frac{3}{4}}$, and hence

$$d(\Lambda^*) \geqslant \tfrac{1}{2}\sqrt{\tfrac{3}{4}}(a+b).$$

The lattice Λ' generated by the points $(0,1,0)$, $(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}, 0)$, $(0, \frac{1}{2}\sqrt{\frac{3}{4}})$ is K^*-admissible. Moreover

$$d(\Lambda') = \tfrac{1}{2}\sqrt{\tfrac{3}{4}}(a+b) \leqslant d(\Lambda^*) = \Delta(K^*).$$

Hence Λ' is a critical lattice of K^*. But Λ' has just ten of its points on the boundary of K^*, viz. $(0,1,0)$, $(\frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}, 0)$, $(\frac{1}{2}a+\frac{1}{2}b, -\frac{1}{2}, 0)$, $(0, \frac{1}{2}, \sqrt{\frac{3}{4}})$, $(0, -\frac{1}{2}, \sqrt{\frac{3}{4}})$ together with their reflections in the origin. It is known, by the result of Swinnerton-Dyer's (**1**) quoted earlier, that any critical lattice of a three-dimensional convex body must have at least twelve of its points on the boundary of the body. This is a contradiction which proves the result.

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CERTAIN THEOREMS OF CAYLEY AND ORR TYPE FOR BILATERAL HYPERGEOMETRIC SERIES

By H. S. SHUKLA (*Lucknow*)

[Received 11 November 1957]

1. Introduction

In 1858 Cayley (**4**) published, without proof, the theorem: *If*

$$(1-z)^{A+B-C}\,{}_2F_1\left[\begin{matrix}2A,\,2B;\\ 2C\end{matrix}\,z\right]=\sum_{n=0}^{\infty}a_n z^n,$$

then

$${}_2F_1\left[\begin{matrix}A,\,B;\\ \frac{1}{2}+C\end{matrix}\,z\right]{}_2F_1\left[\begin{matrix}C-A,\,C-B;\\ \frac{1}{2}+C\end{matrix}\,z\right]=\sum_{n=0}^{\infty}\frac{(C)_n}{(\frac{1}{2}+C)_n}a_n z^n, \qquad (1.1)$$

and, in 1899, Cayley's theorem was proved by Orr (**5**) who obtained some additional theorems of a similar type, Again Bailey (**2**) in 1935, Burchnall and Chaundy (**3**) in 1949, and P. Henrici (**6**) in 1955 deduced further results of the above type. In this paper I obtain certain theorems involving products of bilateral hypergeometric series. Some of these theorems are the generalizations of known results cited above; others are believed to be new.

The following notation has been used throughout the paper:

$$(a)_n=a(a+1)(a+2)\ldots(a+n-1), \qquad (a)_0=1,$$

$$(a)_{-n}=(-1)^n/(1-a)_n,$$

$${}_rH_r\left[\begin{matrix}a_1,\,a_2,\ldots,\,a_r;\\ b_1,\,b_2,\ldots,\,b_r\end{matrix}\,z\right]=\sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_r)_n}z^n,$$

$$\Gamma\left[\begin{matrix}a_1,\,a_2,\ldots,\,a_r;\\ b_1,\,b_2,\ldots,\,b_s\end{matrix}\right]=\frac{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_r)}{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_s)},$$

and idem$(a;\,b)$ means that the preceding expression is repeated with a and b interchanged.

2. Slater (**7**) in 1952 established the relation

$$\Gamma\left[\begin{matrix}1+a_2-a_1,\,1+a_3-a_1,\ldots,\,1+a_M-a_1,\,a_1-a_2,\,a_1-a_3,\ldots,\,a_1-a_M;\\ 1+b_1-a_1,\,1+b_2-a_1,\ldots,\,1+b_M-a_1,\,a_1-c_1,\,a_1-c_2,\ldots,\,a_1-c_M\end{matrix}\right]\times$$

$$\times\,{}_MH_M\left[\begin{matrix}1+c_1-a_1,\,1+c_2-a_1,\ldots,\,1+c_M-a_1;\\ 1+b_1-a_1,\,1+b_2-a_1,\ldots,\,1+b_M-a_1\end{matrix}\right]+$$

$$+\text{idem}(a_1;\,a_2,\,a_3,\ldots,\,a_M)=0. \qquad (2.1)$$

If we first replace M by $M+N$ in (2.1) and then c_{M+N} and b_{M+N} by $c_{M+N}-n$ and $b_{M+N}-n$ respectively, we obtain,

$$\Gamma\left[\begin{matrix}1+a_2-a_1,\ 1+a_3-a_1,\ldots,\ 1+a_{M+N}-a_1,\ a_1-a_2,\ a_1-a_3,\ldots,\ a_1-a_{M+N};\\ 1+b_1-a_1,\ 1+b_2-a_1,\ldots,\ 1+b_{M+N}-a_1,\ a_1-c_1,\ a_1-c_2,\ldots,\ a_1-c_{M+N}\end{matrix}\right]\times$$

$$\times\frac{(a_1-b_{M+N})_n}{(a_1-c_{M+N})_n}{}_{M+N}H_{M+N}\left[\begin{matrix}1+c_1-a_1,\ldots,\ 1+c_{M+N-1}-a_1,\ 1+c_{M+N}-a_1-n;\\ 1+b_1-a_1,\ldots,\ 1+b_{M+N-1}-a_1,\ 1+b_{M+N}-a_1-n\end{matrix}\right]$$

$$+\text{idem}(a_1;\ a_2,\ a_3,\ldots,\ a_{M+N})=0. \quad (2.2)$$

Taking now $M=2$, $N=1$, $a_1=b_1=1$, and $c_3=0$ in (2.2), we get

$$(1-a_3)\Gamma\left[\begin{matrix}1+a_3-a_2,\ a_2-a_3;\\ 1+b_2-a_2,\ 1+b_3-a_2,\ a_2-c_1,\ a_2-c_2\end{matrix}\right]\frac{(a_2-b_3)_n}{(a_2)_n}\times$$

$$\times{}_3H_3\left[\begin{matrix}1+c_1-a_2,\ 1+c_2-a_2,\ 1-a_2-n;\\ 2-a_2,\ 1+b_2-a_2,\ 1+b_3-a_2-n\end{matrix}\right]+\text{idem}(a_2;\ a_3)$$

$$=\Gamma\left[\begin{matrix}a_2,\ a_3,\ 2-a_2,\ 2-a_3;\\ b_2,\ b_3,\ 1-c_1,\ 1-c_2\end{matrix}\right]\frac{(1-b_3)_n}{(1)_n}{}_3F_2\left[\begin{matrix}c_1,\ c_2,\ -n;\\ b_2,\ b_3-n\end{matrix}\right]. \quad (2.3)$$

Next replace c_1, c_2, b_2, a_2, a_3 respectively by $2A$, $2B$, $2C$, D, E and take $b_3=\frac{1}{2}+A+B-C$ in (2.3). Then use first § 4.5 (1), with $c=\frac{1}{2}(1+a)$, and then § 7.2 (1) of Bailey (1) after transforming the terminating ${}_3F_2$ on the right with the help of the transformation

$${}_3F_2\left[\begin{matrix}a,\ b,\ -n;\\ e,\ f\end{matrix}\right]=\frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}{}_3F_2\left[\begin{matrix}1+a+b-e-f-n,\ a,\ -n;\\ 1+a-e-n,\ 1+a-f-n\end{matrix}\right]. \quad (2.4)$$

We then get the transformation

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-E;\\ 1+2C-D,\ \frac{3}{2}+A+B-C-D,\ D-2A,\ D-2B\end{matrix}\right]\times$$

$$\times\frac{(D-\frac{1}{2}-A-B+C)_n}{(D)_n}\times$$

$$\times{}_3H_3\left[\begin{matrix}1+2A-D,\ 1+2B-D,\ 1-D-n;\\ 2-D,\ 1+2C-D,\ \frac{3}{2}+A+B-C-D-n\end{matrix}\right]+\text{idem}(D;\ E)$$

$$=\Gamma\left[\begin{matrix}D,\ E,\ 2-D,\ 2-E;\\ 2C,\ \frac{1}{2}+A+B-C,\ 1-2A,\ 1-2B\end{matrix}\right]\frac{(\frac{1}{2}+C-A)_n(\frac{1}{2}+C-B)_n}{(1)_n(\frac{1}{2}+C)_n}\times$$

$$\times{}_4F_3\left[\begin{matrix}A,\ B,\ -C-n,\ -n;\\ C,\ \frac{1}{2}+A-C-n,\ \frac{1}{2}+B-C-n\end{matrix}\right]. \quad (2.5)$$

Now with the help of (2.5), when we compare the coefficients of z^n, we obtain the theorem:

THEOREM I. *If*

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-E;\\ 1+2C-D,\ \frac{3}{2}+A+B-C-D,\ D-2A,\ D-2B\end{matrix}\right]\times$$
$$\times {}_1H_1\left[\begin{matrix}D-\frac{1}{2}-A-B+C;\\ D\end{matrix}\ z\right]{}_2H_2\left[\begin{matrix}1+2A-D,\ 1+2B-D;\\ 2-D,\ 1+2C-D\end{matrix}\ z\right]+$$
$$+\text{idem}(D;\ E)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix}D,\ E,\ 2-D,\ 2-E;\\ 2C,\ \frac{1}{2}+A+B-C,\ 1-2A,\ 1-2B\end{matrix}\right]{}_2F_1\left[\begin{matrix}A,\ B;\\ C\end{matrix}\ z\right]\times$$
$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}+C-A,\ \frac{1}{2}+C-B;\\ 1+C\end{matrix}\ z\right]$$
$$=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}+C)_n}{(1+C)_n}a_n z^n,\quad \textit{where } |z|=1.$$

Again, in (2.5), replace A, C by $A-\frac{1}{2}$, $C-\frac{1}{2}$ respectively, and then transform the terminating ${}_4F_3$ on the right by § 7.2 (1) of Bailey (**1**). We get another relation with the help of which, when we compare the coefficients of z^n, we obtain the theorem:

THEOREM II. *If*

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-E;\\ 2C-D,\ \frac{3}{2}+A+B-C-D,\ 1+D-2A,\ D-2B\end{matrix}\right]\times$$
$$\times {}_1H_1\left[\begin{matrix}D-\frac{1}{2}-A-B+C;\\ D\end{matrix}\ z\right]{}_2H_2\left[\begin{matrix}2A-D,\ 1+2B-D;\\ 2-D,\ 2C-D\end{matrix}\ z\right]+$$
$$+\text{idem}(D;\ E)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix}D,\ E,\ 2-D,\ 2-E;\\ 2C-1,\ \frac{1}{2}+A+B-C,\ 2-2A,\ 1-2B\end{matrix}\right]\times$$
$$\times {}_2F_1\left[\begin{matrix}A,\ B;\\ C\end{matrix}\ z\right]{}_2F_1\left[\begin{matrix}\frac{1}{2}+C-A,\ C-B-\frac{1}{2};\\ C\end{matrix}\ z\right]$$
$$=\sum_{n=0}^{\infty}\frac{(C-\frac{1}{2})_n}{(C)_n}a_n z^n,\quad \textit{where } |z|=1.$$

If we put $D=1$ in Theorems I and II, we get, as their special cases, two theorems due to Orr given in § 10.1 of Bailey (**1**).

Again, replace c_1, c_2, b_2, a_2, a_3 respectively by $2A$, $2B$, $2C$, D, E, take $b_3 = 1+A+B-C$ in (2.3), and then transform the terminating ${}_3F_2$ on the right into a terminating Saalschützian ${}_4F_3$ with the help of the known result given in § 10.1 (1) of Bailey (**1**). We get

$$(1-E)\Gamma\left[\begin{matrix} 1+E-D, D-E; \\ 1+2C-D, 2+A+B-C-D, D-2A, D-2B \end{matrix}\right] \times$$
$$\times \frac{(D-1-A-B+C)_n}{(D)_n} \times$$
$$\times {}_3H_3\left[\begin{matrix} 1+2A-D, 1+2B-D, 1-D-n; \\ 2-D, 1+2C-D, 2+A+B-C-D-n \end{matrix}\right] + \text{idem}(D; E)$$
$$= \Gamma\left[\begin{matrix} D, E, 2-D, 2-E; \\ 2C, 1+A+B-C, 1-2A, 1-2B \end{matrix}\right] \frac{(C-A)_n (C-B)_n}{(1)_n (C)_n} \times$$
$$\times {}_4F_3\left[\begin{matrix} A, B, \tfrac{1}{2}-C-n, -n; \\ \tfrac{1}{2}+C, 1+A-C-n, 1+B-C-n \end{matrix}\right]. \quad (2.6)$$

Now the transformation (2.6) gives, on comparison of coefficients of z^n, the theorem:

THEOREM III. *If*

$$(1-E)\Gamma\left[\begin{matrix} 1+E-D, D-E; \\ 1+2C-D, 2+A+B-C-D, D-2A, D-2B \end{matrix}\right] \times$$
$$\times {}_1H_1\left[\begin{matrix} C-A-B-1+D; \\ D \end{matrix} z\right] {}_2H_2\left[\begin{matrix} 1+2A-D, 1+2B-D; \\ 2-D, 1+2C-D \end{matrix} z\right] +$$
$$+ \text{idem}(D; E) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

then

$$\Gamma\left[\begin{matrix} D, E, 2-D, 2-E; \\ 2C, 1+A+B-C, 1-2A, 1-2B \end{matrix}\right] {}_2F_1\left[\begin{matrix} A, B; \\ \tfrac{1}{2}+C \end{matrix} z\right] {}_2F_1\left[\begin{matrix} C-A, C-B; \\ \tfrac{1}{2}+C \end{matrix} z\right]$$
$$= \sum_{n=0}^{\infty} \frac{(C)_n}{(C+\tfrac{1}{2})_n} a_n z^n, \quad \textit{where } |z| = 1.$$

If we put $D = 1$ in Theorem III, we get Cayley's theorem (1.1). Bailey [(**2**) Theorems I–III] has obtained several other theorems, the type of those given by Cayley and Orr involving products of a ${}_1F_0$ and a ${}_3F_2$. It may be remarked that with the above technique after taking $M = 3$, $N = 1$, $a_1 = b_1 = 1$, and $c_4 = 0$ in (2.2) and transforming the terminating ${}_4F_3$ in the manner Bailey has transformed them to prove his theorems, we can get the generalizations of Bailey's theorems. However, in this case there will be the sum of three products of bilateral

hypergeometric series on the left instead of the sum of two as in the above Theorems I–III.

3. In this section I obtain a generalization of Henrici's theorem (**6**). We replace c_1, c_2, b_3, a_2, a_3 respectively by $1+K+B$, $1+K$, $1+K-A$, D, E, take $b_2 = 1+B$ in (2.3), and then transform the terminating ${}_3F_2$ on the right into a terminating Saalschützian ${}_4F_3$ by using first § 4.5 (1) and then § 7.2 (1) of Bailey (**1**). We get

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-E;\\ 2+B-D,\ 2+K-A-D,\ D-1-K-B,\ D-1-K\end{matrix}\right]\times$$
$$\times\frac{(D-1-K+A)_n}{(D)_n}\times$$
$$\times{}_3H_3\left[\begin{matrix}2+K+B-D,\ 2+K-D,\ 1-D-n;\\ 2-D,\ 2+B-D,\ 2+K-A-D-n\end{matrix}\right]+\text{idem}(D;\,E)$$
$$=\Gamma\left[\begin{matrix}D,\ E,\ 2-D,\ 2-E;\\ 1+B,\ 1+K-A,\ -K-B,\ -K\end{matrix}\right]\frac{(A-K)_n(A-K+n)_n}{(1)_n(1+A+B+n)_n}\times$$
$$\times{}_4F_3\left[\begin{matrix}\tfrac{1}{2}(1+K+B),\ \tfrac{1}{2}(2+K+B),\ -A-n,\ -n;\\ 1+B,\ \tfrac{1}{2}(2+K-A)-n,\ \tfrac{1}{2}(1+K-A)-n\end{matrix}\right]. \quad (3.1)$$

Now with the help of (3.1), when we compare the coefficients of z^n, we deduce the theorem

THEOREM IV. *If*

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-E;\\ 2+B-D,\ 2+K-A-D,\ D-1-K-B,\ D-1-K\end{matrix}\right]\times$$
$$\times{}_1H_1\left[\begin{matrix}D-1-K+A;\\ D\end{matrix}\ z\right]{}_2H_2\left[\begin{matrix}2+K+B-D,\ 2+K-D;\\ 2-D,\ 2+B-D\end{matrix}\ z\right]+$$
$$+\text{idem}(D;\,E)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix}D,\ E,\ 2-D,\ 2-E;\\ 1+B,\ 1+K-A,\ -K-B,\ -K\end{matrix}\right]{}_2F_1\left[\begin{matrix}\tfrac{1}{2}(1+K+B),\ \tfrac{1}{2}(2+K+B);\\ 1+B\end{matrix}\ z\right]\times$$
$$\times{}_2F_1\left[\begin{matrix}\tfrac{1}{2}(A-K),\ \tfrac{1}{2}(1+A-K);\\ 1+A\end{matrix}\ z\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1+A+B+n)_n}{2^{2n}(1+A)_n}a_n z^n \quad \textit{where } |z|=1.$$

If we put $D=1$ in Theorem IV, we get the corollary:

COROLLARY. *If*

$$(1-z)^{K-A}\,{}_2F_1\left[\begin{matrix}1+K+B,\ 1+K;\\ 1+B\end{matrix}\ z\right]=\sum_{n=0}^{\infty}a_n z^n,$$

then

$${}_2F_1\left[\begin{matrix}\frac{1}{2}(1+K+B),\ \frac{1}{2}(2+K+B);\\ 1+B\end{matrix}\ z\right]{}_2F_1\left[\begin{matrix}\frac{1}{2}(A-K),\ \frac{1}{2}(1+A-K);\\ 1+A\end{matrix}\ z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+A+B+n)_n}{2^{2n}(1+A)_n}a_n z^n,\quad \textit{where } |z|<1,$$

which is Henrici's theorem given in (6).

4. So far I have deduced the generalization of certain known theorems. In this section I obtain certain theorems which are believed to be new. If we first put $M=3$, $N=1$, $a_1=b_1=1$, $c_4=0$, $a_2=b_2=1+c_1-c_2$, and $b_3=1+c_1-c_3$ in (2.2), we get a relation between two non-terminating nearly-poised ${}_4H_4$, a non-terminating nearly-poised ${}_4F_3$, and a terminating nearly-poised ${}_4F_3$. If we put $c_2=1+\frac{1}{2}c_1$ in this new relation and replace c_1, c_3, b_4, a_3, a_4 respectively by A, B, W, D, E, we obtain the transformation,

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-\frac{1}{2}A,\ D-E;\\ 2+A-B-D,\ 1+W-D,\ D-A,\ D-1-\frac{1}{2}A,\ D-B\end{matrix}\right]\frac{(D-W)_n}{(D)_n}\times$$

$$\times\,{}_4H_4\left[\begin{matrix}1+A-D,\ 2+\frac{1}{2}A-D,\ 1+B-D,\ 1-D-n;\\ 2-D,\ 1+\frac{1}{2}A-D,\ 2+A-B-D,\ 1+W-D-n\end{matrix}\right]+\text{idem}(D;\,E)$$

$$=\Gamma\left[\begin{matrix}D,\ E,\ 1-\frac{1}{2}A,\ 2-D,\ 2-E;\\ 1+A-B,\ W,\ 1-A,\ -\frac{1}{2}A,\ 1-B\end{matrix}\right]\frac{(1-W)_n}{(1)_n}\times$$

$$\times\,{}_4F_3\left[\begin{matrix}A,\ 1+\frac{1}{2}A,\ B,\ -n;\\ \frac{1}{2}A,\ 1+A-B,\ W-n\end{matrix}\right].\qquad(4.1)$$

Now using first § 4.5 (1), with $c=1+\frac{1}{2}a$, and then § 7.2 (1) of Bailey (1) to transform the ${}_4F_3$ on the right of (4.1), we obtain,

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-\frac{1}{2}A,\ D-E;\\ 2+A-B-D,\ 1+W-D,\ D-A,\ D-1-\frac{1}{2}A,\ D-B\end{matrix}\right]\frac{(D-W)_n}{(D)_n}\times$$

$$\times\,{}_4H_4\left[\begin{matrix}1+A-D,\ 2+\frac{1}{2}A-D,\ 1+B-D,\ 1-D-n;\\ 2-D,\ 1+\frac{1}{2}A-D,\ 2+A-B-D,\ 1+W-D-n\end{matrix}\right]+\text{idem}(D;\,E)$$

$$=\Gamma\left[\begin{matrix}D,\ E,\ 1-\frac{1}{2}A,\ 2-D,\ 2-E;\\ 1+A-B,\ W,\ 1-A,\ -\frac{1}{2}A,\ 1-B\end{matrix}\right]\frac{(1+A-W)_n(-W)_{2n}}{(1)_n(1+A-W)_{2n}}\times$$

$$\times\,{}_4F_3\left[\begin{matrix}\frac{1}{2}(1+A),\ 1+\frac{1}{2}A,\ W-B-n,\ -n;\\ 1+A-B,\ 1+\frac{1}{2}W-n,\ \frac{1}{2}(1+W)-n\end{matrix}\right].\qquad(4.2)$$

Hence, with the help of (4.2), when we compare the coefficients of z^n, we obtain the theorem:

THEOREM V. *If*

$$(1-E)\Gamma\left[\begin{matrix}1+E-D,\ D-\frac{1}{2}A,\ D-E;\\ 2+A-B-D,\ 1+W-D,\ D-A,\ D-1-\frac{1}{2}A,\ D-B\end{matrix}\right]\times$$

$$\times {}_1H_1\left[\begin{matrix}D-W;\\ D\end{matrix}\ z\right]{}_3H_3\left[\begin{matrix}1+A-D,\ 2+\frac{1}{2}A-D,\ 1+B-D;\\ 2-D,\ 1+\frac{1}{2}A-D,\ 2+A-B-D\end{matrix}\ z\right]+$$

$$+\text{idem}(D;\ E)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix}D,\ E,\ 1-\frac{1}{2}A,\ 2-D,\ 2-E;\\ 1+A-B,\ W,\ 1-A,\ -\frac{1}{2}A,\ 1-B\end{matrix}\right]{}_2F_1\left[\begin{matrix}\frac{1}{2}(1+A),\ 1+\frac{1}{2}A;\\ 1+A-B\end{matrix}\ z\right]\times$$

$$\times {}_2F_1\left[\begin{matrix}\frac{1}{2}(1-W),\ -\frac{1}{2}W;\\ 1+B-W\end{matrix}\ z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+A-W+n)_n}{2^{2n}(1+B-W)_n}a_n z^n,\quad \textit{where } |z|=1.$$

If we put $D=1$ in Theorem V, we get the corollary:

COROLLARY. *If*

$$(1-z)^{W-1}{}_3F_2\left[\begin{matrix}A,\ 1+\frac{1}{2}A,\ B;\\ \frac{1}{2}A,\ 1+A-B\end{matrix}\ z\right]=\sum_{n=0}^{\infty}a_n z^n,$$

then

$${}_2F_1\left[\begin{matrix}\frac{1}{2}(1+A),\ 1+\frac{1}{2}A;\\ 1+A-B\end{matrix}\ z\right]{}_2F_1\left[\begin{matrix}\frac{1}{2}(1-W),\ -\frac{1}{2}W;\\ 1+B-W\end{matrix}\ z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+A-W+n)_n}{2^{2n}(1+B-W)_n}a_n z^n,\quad \textit{where } |z|<1.$$

This is believed to be a new result.

Again, if we take

$$M=4,\quad N=1,\quad a_1=b_1=1,\quad c_5=0,\quad a_2=b_2=1+c_1-c_2,$$

$$b_3=1+c_1-c_3,\quad b_4=1+c_1-c_4$$

in (2.2) and then in the new relation put

$$c_2=1+\tfrac{1}{2}c_1\quad\text{and}\quad b_5=2c_3+2c_4-c_1,$$

we get a relation between three nearly-poised non-terminating ${}_5H_5$ and a terminating nearly-poised Saalschützian ${}_5F_4$. Now replace c_1, c_3, c_4, a_3, a_4, a_5 respectively by A, B, C, D, E, G and use first § 4.5 (2) (after

making the nearly-poised ${}_5F_4$ a Saalschützian to reduce the Saalschützian ${}_5F_4$ into a Saalschützian ${}_4F_3$) and then § 7.2 (1) of Bailey (**1**). We obtain

$$(1-E)(1-G)\times$$

$$\times\Gamma\left[\begin{matrix}1+E-D,\,1+G-D,\,D-\tfrac{1}{2}A,\,D-E,\,D-G;\\ 2+A-B-D,\,2+A-C-D,\,1+2B+2C-A-D,\\ D-A,\,D-1-\tfrac{1}{2}A,\,D-B,\,D-C\end{matrix}\right]\times$$

$$\times\frac{(D+A-2B-2C)_n}{(D)_n}\times$$

$$\times{}_5H_5\left[\begin{matrix}1+A-D,\,2+\tfrac{1}{2}A-D,\,1+B-D,\,1+C-D,\,1-D-n;\\ 2-D,\,1+\tfrac{1}{2}A-D,\,2+A-B-D,\,2+A-C-D,\\ 1+2B+2C-A-D-n\end{matrix}\right]+$$

$$+\text{idem}(D;\,E,\,G)$$

$$=\Gamma\left[\begin{matrix}D,\,E,\,G,\,1-\tfrac{1}{2}A,\,2-D,\,2-E,\,2-G;\\ 1+A-B,\,1+A-C,\,2B+2C-A,\,1-A,\,-\tfrac{1}{2}A,\,1-B,\,1-C\end{matrix}\right]\times$$

$$\times\frac{(1+2A-2B-2C)_n}{(1+A-B)_n}\,\frac{(\tfrac{1}{2}A-B)_n(\tfrac{1}{2}+\tfrac{1}{2}A-B-C)_n}{(1)_n(\tfrac{1}{2}+A-B-C)_n}\times$$

$$\times{}_4F_3\left[\begin{matrix}1+\tfrac{1}{2}A,\,\tfrac{1}{2}(1+A)-C,\,2B+C-A-n,\,-n;\\ 1+A-C,\,1+B-\tfrac{1}{2}A-n,\,\tfrac{1}{2}+B+C-\tfrac{1}{2}A-n\end{matrix}\right].\qquad(4.3)$$

Hence with the help of (4.3), when we compare the coefficients of z^n, we obtain the theorem:

THEOREM VI. *If*

$$(1-E)(1-G)\times$$

$$\times\Gamma\left[\begin{matrix}1+E-D,\,1+G-D,\,D-\tfrac{1}{2}A,\,D-E,\,D-G;\\ 2+A-B-D,\,2+A-C-D,\,1+2B+2C-A-D,\\ D-A,\,D-1-\tfrac{1}{2}A,\,D-B,\,D-C\end{matrix}\right]\times$$

$$\times{}_1H_1\left[\begin{matrix}D+A-2B-2C;\\ D\end{matrix}\,z\right]\times$$

$$\times{}_4H_4\left[\begin{matrix}1+A-D,\,2+\tfrac{1}{2}A-D,\,1+B-D,\,1+C-D;\\ 2-D,\,1+\tfrac{1}{2}A-D,\,2+A-B-D,\,2+A-C-D\end{matrix}\,z\right]+$$

$$+\text{idem}(D;\,E,\,G)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix} D, E, G, 1-\frac{1}{2}A, 2-D, 2-E, 2-G; \\ 1+A-B, 1+A-C, 2B+2C-A, 1-A, -\frac{1}{2}A, 1-B, 1-C \end{matrix}\right] \times$$

$$\times {}_2F_1\left[\begin{matrix} 1+\frac{1}{2}A, \frac{1}{2}(1+A)-C; \\ 1+A-C \end{matrix} z\right] {}_2F_1\left[\begin{matrix} \frac{1}{2}A-B, \frac{1}{2}(1+A)-B-C; \\ 1+A-2B-C \end{matrix} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+A-B-C)_n (1+A-B)_n}{(1+2A-2B-2C)_n (1+A-2B-C)_n} a_n z^n, \quad \textit{where } |z| = 1.$$

If we put $D = 1$ in Theorem VI, we get the corollary:

COROLLARY. *If*

$$(1-z)^{2B+2C-A-1} {}_4F_3\left[\begin{matrix} A, 1+\frac{1}{2}A, B, C; \\ \frac{1}{2}A, 1+A-B, 1+A-C \end{matrix} z\right] = \sum_{n=0}^{\infty} a_n z^n,$$

then

$${}_2F_1\left[\begin{matrix} 1+\frac{1}{2}A, \frac{1}{2}(1+A)-C; \\ 1+A-C \end{matrix} z\right] {}_2F_1\left[\begin{matrix} \frac{1}{2}A-B, \frac{1}{2}(1+A)-B-C; \\ 1+A-2B-C \end{matrix} z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+A-B-C)_n (1+A-B)_n}{(1+2A-2B-2C)_n (1+A-2B-C)_n} a_n z^n, \quad \textit{where } |z| < 1.$$

This also is believed to be a new result.

In § 4.5 (2) of Bailey (**1**) the Saalschützian ${}_5F_4$ is reducible to a Saalschützian ${}_4F_3$ when the nearly-poised ${}_5F_4$ on the left is such that in it the sum of the denominator parameters exceeds that of the numerator parameters by two. Hence following the above technique we can also obtain another theorem of the type of Theorem VI. Also we can get two more theorems if, with the help of § 4.5 (1) of Bailey (**1**), we transform the terminating ${}_4F_3$ in the relation involving three nearly-poised ${}_4H_4$ and a terminating nearly-poised ${}_4F_3$ which can be obtained from (2.2). For then the nearly-poised ${}_4F_3$ series in it can be made either Saalschützian or such that in it the sum of the denominator parameters exceeds that of the numerator parameters by two.

5. In this section I obtain a theorem containing more parameters. We put first

$$M = 4, \quad a_1 = b_1 = 1, \quad c_4 = -n, \quad b_2 = 1+c_1-c_2, \quad b_3 = 1+c_1-c_3$$

in (2.1) and then in the new relation replace c_1, c_2, c_3, b_4, a_2, a_3, a_4 respectively by $A-n$, B, C, W, D, E, G. We get a relation between three non-terminating nearly-poised ${}_4H_4$ and a terminating nearly-poised ${}_4F_3$ of the second kind. Now using § 4.5 (1) of Bailey (**1**) to

transform the terminating ${}_4F_3$ into a terminating ${}_5F_4$, we obtain

$$(1-E)(1-G)\times$$
$$\times\Gamma\left[\begin{matrix}1+E-D,\ 1+G-D,\ D-E,\ D-G;\\ 2+A-B-D,\ 2+A-C-D,\ 1+W-D,\ D-A,\ D-B,\ D-C\end{matrix}\right]\times$$
$$\times\frac{(D+B-A-1)_n(D+C-A-1)_n}{(D)_n(D-A)_n}\times$$
$$\times{}_4H_4\left[\begin{matrix}1+A-D-n,\ 1+B-D,\ 1+C-D,\ 1-D-n;\\ 2-D,\ 2+A-B-D-n,\ 2+A-C-D-n,\ 1+W-D\end{matrix}\right]+$$
$$+\text{idem}(D;\ E,\ G)$$
$$=\Gamma\left[\begin{matrix}D,\ E,\ G,\ 2-D,\ 2-E,\ 2-G;\\ 1+A-B,\ 1+A-C,\ W,\ 1-A,\ 1-B,\ 1-C\end{matrix}\right]\times$$
$$\times\frac{(B-A)_n(C-A)_n(W-A+n)_n}{(1)_n(1-A)_n(W)_n}\times$$
$$\times{}_5F_4\left[\begin{matrix}1+A-W-n,\ \frac{1}{2}(A-n),\ \frac{1}{2}(1+A-n),\ 1+A-B-C-n,\ -n;\\ 1+A-B-n,\ 1+A-C-n,\ \frac{1}{2}(1+A-W)-n,\ 1+\frac{1}{2}(A-W)-n\end{matrix}\right].$$
(5.1)

The transformation (5.1), when we compare the coefficients of z^n, leads to the theorem:

THEOREM VII. *If*

$$\Gamma\left[\begin{matrix}1+E-D,\ 1+G-D,\ D-E,\ D-G;\\ 2+A-B-D,\ 2+A-C-D,\ 1+W-D,\ D-A,\ D-B,\ D-C\end{matrix}\right]\times$$
$$\times(1-E)(1-G){}_2H_2\left[\begin{matrix}1+B-D,\ 1+C-D;\\ 2-D,\ 1+W-D\end{matrix}\ z\right]\times$$
$$\times{}_2H_2\left[\begin{matrix}D+B-A-1,\ D+C-A-1;\\ D,\ D-A\end{matrix}\ z\right]+$$
$$+\text{idem}(D;\ E,\ G)=\sum_{n=-\infty}^{\infty}a_n z^n,$$

then

$$\Gamma\left[\begin{matrix}D,\ E,\ G,\ 2-D,\ 2-E,\ 2-G;\\ 1+A-B,\ 1+A-C,\ W,\ 1-A,\ 1-B,\ 1-C\end{matrix}\right](1-z)^{-A}\times$$
$$\times{}_4F_3\left[\begin{matrix}B-A,\ C-A,\ \frac{1}{2}(W-A),\ \frac{1}{2}(1+W-A);\\ W-A,\ B+C-A,\ 1-A\end{matrix}\ 4z(1-z)\right]$$
$$=\sum_{n=0}^{\infty}\frac{(W)_n}{(B+C-A)_n}a_n z^n,\quad \textit{where } |z|=1.$$

If we put $D=1$ in Theorem VII, we get the corollary:

COROLLARY. *If*

$$ {}_2F_1\left[\begin{matrix} B, C; \\ W \end{matrix} z\right] {}_2F_1\left[\begin{matrix} B-A, C-A; \\ 1-A \end{matrix} z\right] = \sum_{n=0}^{\infty} a_n z^n, $$

then

$$ (1-z)^{-A} {}_4F_3\left[\begin{matrix} B-A, C-A, \frac{1}{2}(W-A), \frac{1}{2}(1+W-A); \\ W-A, B+C-A, 1-A \end{matrix} 4z(1-z)\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(W)_n}{(B+C-A)_n} a_n z^n, \quad \textit{where } |z| < 1. $$

The above result also appears to be new. It may be remarked here that bilateral analogues of Theorems IV, V, VI of Bailey (**2**) can be easily obtained by using the above technique and suitable substitutions. Also as special cases of the above theorems one can deduce many interesting results concerning products of hypergeometric series.

6. Lastly, I deduce a transformation with the help of which we can obtain theorems which have on the left only one product of bilateral series instead of the sum of two products of bilateral series, but in this case they will be the generalizations of particular cases of the theorems of Cayley, Orr, and Henrici.

If in (2.1) we first put $b_{M+1} = a_{M+1}$, $b_{M+2} = a_{M+2}, \ldots, b_{M+N} = a_{M+N}$ after replacing M by $M+N$ in it and then in the new relation put $a_{M+1} = c_1 - n$, $a_{M+2} = c_2 - n, \ldots, a_{M+N} = c_N - n$, and $c_{M+1} = b_1 - 1 - n$, $c_{M+2} = b_2 - 1 - n, \ldots, c_{M+N} = b_N - 1 - n$, we obtain

$$ \Gamma\left[\begin{matrix} a_1-a_2,\ldots, a_1-a_M, a_1-c_1,\ldots, a_1-c_N, 1+a_2-a_1,\ldots, 1+a_M-a_1; \\ a_1-c_1,\ldots, a_1-c_M, 1+a_1-b_1,\ldots, 1+a_1-b_N, 1+b_1-a_1,\ldots, 1+b_M-a_1 \end{matrix}\right] \times $$

$$ \times \frac{(a_1-c_1)_n \ldots (a_1-c_N)_n}{(1+a_1-b_1)_n \ldots (1+a_1-b_N)_n} \times $$

$$ \times {}_{M+N}H_{M+N}\left[\begin{matrix} 1+c_1-a_1,\ldots, 1+c_M-a_1, b_1-a_1-n,\ldots, b_N-a_1-n; \\ 1+b_1-a_1,\ldots, 1+b_M-a_1, 1+c_1-a_1-n,\ldots, 1+c_N-a_1-n \end{matrix}\right] + $$

$$ + \text{idem}(a_1; a_2, a_3, \ldots, a_M) = 0. \quad (6.1) $$

Now, if we put $M = 2$, $N = 1$, and $a_1 = b_1 = 1$ in (6.1) and then transform the terminating ${}_3F_2$ into a terminating ${}_4F_3$ with the help of known results after making suitable substitutions, we get the identities required for proving theorems which are generalizations of particular cases of Cayley, Orr, and Henrici's theorems and in these cases there will be only one product of bilateral series on the left. If we use (6.1) for getting generalizations of particular cases of Bailey's theorems, we

shall have the sum of two products of bilateral series on the left instead of the sum of three products.

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THE ASYMPTOTIC BEHAVIOUR OF THE GENERATING FUNCTIONS OF PARTITIONS OF MULTI-PARTITES

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1. Introduction

SEVERAL authors (**1**, **5**, **9**) have recently found asymptotic formulae for the number of partitions of a multi-partite number with large components. A multi-partite number is a j-dimensional vector with positive integral components. In every case, the method is an adaptation to $j > 1$ of the celebrated Hardy–Ramanujan method (**2**) for $j = 1$, sometimes in the modified form due to Ingham (**3**). This involves the study of generating functions, a simple example of which is

$$f(x_1,\ldots,x_j) = \prod_{h_1,\ldots,h_j=1}^{\infty} (1-e^{-h_1x_1-\ldots-h_jx_j})^{-1} = \sum_{n_1,\ldots,n_j=0}^{\infty} p(n_1,\ldots,n_j)e^{-n_1x_1-\ldots-n_jx_j}, \tag{1.1}$$

where $\mathrm{re}(x_l) > 0$ for every l and $p(0,\ldots,0) = 1$. The first step is to obtain a suitable asymptotic approximation to f as $x_1,\ldots, x_j \to 0$. So far, for $j > 1$, we have found and used only very crude approximations of this kind. Here I find a much closer approximation to f for small x_l. I have used these results elsewhere (**11**) to obtain an asymptotic formula for $p(n_1, n_2)$ valid under wider conditions on the ratio n_1/n_2 than those of (**1**) and (**9**) (i.e. when n_1, n_2 are not of the same order) and similar results could be found for $j > 2$. Again we could use the results to find further terms of the asymptotic expansion of $p(n_1,\ldots, n_j)$.

For practical purposes only the first term or so of the expansion of $p(n_1,\ldots, n_j)$ is of importance. There is, however, another point of considerable theoretical interest. The extreme closeness of approximation to $f(x_1)$ obtained by Hardy and Ramanujan was an essential element in the striking success of their method. In other partition problems, the generating function is rarely susceptible of such close approximation. The case of kth-power partitions (**8**) is one of the very few examples of this being partially possible, that of plane partitions (**7**) an example of it being apparently impossible. It seems worth considering, therefore, whether any of the multi-partite generating functions are capable of anything approaching the closeness of approximation

valid in the case $j = 1$. The answer appears to be that it is possible only for generating functions for certain somewhat artificial types of weighted partition, and then in a less simple form than for $j = 1$.

We suppose $x_1, \ldots, x_j$ small but impose no condition on their relative order of smallness. We take

$$\epsilon_1 > 0, \qquad |\arg x_l| < \tfrac{1}{2}\pi - \epsilon_1$$

and write x for the x_l with greatest modulus, i.e.

$$|x| = \max |x_l|.$$

We write also $L = x_1 \ldots x_j$ and

$$F = F_j = F(x_1, x_2, \ldots, x_j) = -\sum_{h_1, \ldots, h_j = 1}^{\infty} \log(1 - e^{-h_1 x_1 - \ldots - h_j x_j})$$

$$= \sum_{r=1}^{\infty} \frac{1}{r} \sum_{h_1, \ldots, h_j = 1}^{\infty} e^{-r h_1 x_1 - r h_2 x_2 - \ldots - r h_j x_j} = \sum_{r=1}^{\infty} \omega(r),$$

where

$$\omega(t) = \frac{1}{t} \prod_{l=1}^{j} \frac{1}{(e^{x_l t} - 1)}.$$

For $|u| < 2\pi$, we have

$$\frac{u}{e^u - 1} = 1 - \tfrac{1}{2}u + \sum_{h=1}^{\infty} \frac{(-1)^{h-1} B_h u^{2h}}{(2h)!},$$

where

$$B_h = \frac{2(2h)!\, \zeta(2h)}{(2\pi)^{2h}}$$

is the hth Bernoulli number. Hence, for $|t| < 2\pi / |x|$, we have

$$L t^{j+1} \omega(t) = \prod_{l=1}^{j} \left(1 - \tfrac{1}{2} x_l t + \sum_{h=1}^{\infty} \frac{(-1)^{h-1} B_h x_l^{2h} t^{2h}}{(2h)!}\right) = \sum_{m=0}^{\infty} R_m t^m, \quad (1.2)$$

where $R_m = R_m(x_1, \ldots, x_j)$ is a symmetric homogeneous polynomial of degree m in $x_1, \ldots, x_j$. We write

$$S_1(t) = \sum_{m=0}^{j-1} R_m t^{m-j-1}, \qquad S(t) = S_1(t) + R_j t^{-1}$$

and

$$G(t) = L\omega(t) - S(t),$$

so that

$$G(t) = \sum_{m=j+1}^{\infty} R_m t^{m-j-1}$$

whenever $|xt| < 2\pi$. We write also

$$H = H(x_1, \ldots, x_j) = R_j \gamma + \int_0^{\infty} \left\{ L\omega(t) - S_1(t) - \frac{R_j}{e^t - 1} \right\} dt$$

and can verify that

$$H = Ld(x_1, \ldots, x_j) - j^{-1} R_j \log L,$$

where $d(x_1,\dots,x_j)$ is homogeneous of degree 0 in $x_1,\dots,x_j$, i.e. d depends only on the ratios of the x_l. Methods to evaluate $d(x_1,x_2)$ are given in (**10**); my colleague Mr. M. M. Robertson has extended these to values of j greater than 2.

In what follows C is a positive number, not always the same at each occurrence, which may depend on ϵ_1, ϵ_2, j, k, l, K, but is independent of t, r, m, n, R, V, $x_1,\dots,x_j$. The $O(\ )$ notation refers to the passage of $|x|$ to zero and the constant implied is a C.

We prove first the theorem:

THEOREM 1. *Under the conditions stated,*

$$LF = \sum_{k=0}^{j-1} \zeta(j+1-k)R_k + H(x_1,\dots,x_j) - \tfrac{1}{2}R_{j+1} + \\ + \sum_{s=1}^{K-1} \frac{(-1)^s B_s R_{2s+j}}{2s} + O(x^{2K+j}). \qquad (1.3)$$

If we use the well-known values of $\zeta(-k)$ [see (**6**) 19], we can write (1.3) more compactly as

$$LF = H + \sum_{\substack{k=0 \\ k\neq j}}^{2K+j-1} \zeta(j+1-k)R_k + O(x^{2K+j}).$$

From Theorem 1 with $j=1$ and with $j=2$, we can deduce the formula for the approximation to the generating function

$$\prod_{h=1}^{\infty}(1-X^h)^{-h}$$

of plane partitions found by a different method in (**7**).

2. Proof of Theorem 1

We require the following lemma:

LEMMA. *As $t\to\infty$ through real, positive values, $G(t)\to 0$ and $G^{(k)}(t)\to 0$ for all $k\geqslant 1$. Again*

$$\int_0^{\infty} |G^{(k)}(t)|\,dt < Cx^{j+k} \quad (k\geqslant 1).$$

Clearly $G(t)\to 0$ as $t\to\infty$. We have

$$\frac{d}{dt}\left(\frac{1}{(e^{x_l t}-1)^{\alpha}}\right) = \frac{-\alpha x_l e^{x_l t}}{(e^{x_l t}-1)^{\alpha+1}} = -\frac{\alpha x_l}{(e^{x_l t}-1)^{\alpha}} - \frac{\alpha x_l}{(e^{x_l t}-1)^{\alpha+1}}.$$

Hence $L\omega^{(k)}(t)$ is the sum of not more than C terms of the form

$$\Lambda = \frac{B}{t^{a+1}} \prod_{l=1}^{j} \frac{x_l^{\beta_l}}{(e^{x_l t}-1)^{\alpha_l}},$$

where $|B| < C$, $a + \sum \beta_l = k+j$, $\beta_l \geqslant \alpha_l \geqslant 1$. Clearly $\omega^{(k)}(t) \to 0$ as $t \to \infty$. But, if

$$t \geqslant t_0 = \tfrac{1}{2}\pi/|x|,$$

we have

$$|e^{x_l t} - 1| > C|x_l|t, \qquad |\Lambda| \leqslant Ct^{-1-a-\sum \alpha_l} \prod |x_l|^{\beta_l - \alpha_l},$$

and so

$$\int_{t_0}^{\infty} |\Lambda|\, dt \leqslant C|x|^{a+\sum \beta_l} = C|x|^{k+j}.$$

Also, for $t \geqslant t_0$,

$$|S^{(k)}(t)| \leqslant C \sum_{m=0}^{j} |x^m| t^{m-k-j-1} \leqslant C|x^j| t^{-k-1} \to 0$$

as $t \to \infty$. Hence $G^{(k)}(t) \to 0$ as $t \to \infty$ and

$$\int_{t_0}^{\infty} |G^{(k)}(t)|\, dt \leqslant \sum \int_{t_0}^{\infty} |\Lambda|\, dt + \int_{t_0}^{\infty} |S^{(k)}(t)|\, dt \leqslant C|x^{k+j}|.$$

Putting $t = \pi/|x|$ in (1.2), we have

$$\sum_{m=0}^{\infty} |R_m| \pi^m |x|^{-m} \leqslant \prod_{l=1}^{j} \left\{1 + \tfrac{1}{2}\pi + \sum_{k=1}^{\infty} \frac{B_k \pi^{2k}}{(2k)!}\right\} = C,$$

and so

$$|R_m| < C|x|^m \pi^{-m}.$$

Hence, for $0 \leqslant t \leqslant t_0$,

$$|G^{(k)}(t)| \leqslant \sum_{m=j+k+1}^{\infty} (m-j-1)\ldots(m-j-k)|R_m| t^{m-j-k-1},$$

$$\int_0^{t_0} |G^{(k)}(t)|\, dt \leqslant \sum_{m=j+k+1}^{\infty} (m-j-1)\ldots(m-j-k+1)|R_m| t_0^{m-j-k}$$

$$\leqslant C|x|^{j+k} \sum_{m=j+k+1}^{\infty} (m-j-1)\ldots(m-j-k+1) 2^{j+k-m}$$

$$= C|x|^{j+k}.$$

This completes the proof of the lemma.

For real, non-negative t we write

$$P_1(t) = t - [t] - \tfrac{1}{2},$$

and define $P_k(t)$ for $k > 1$ by

$$P_k'(t) = P_{k-1}(t), \qquad P_{2k}(0) = \frac{(-1)^{k-1} B_k}{(2k)!}, \qquad P_{2k+1}(0) = 0.$$

Then [by (4) 524] every $P_k(t)$ has period 1 and is bounded. If R is any large positive integer,

$$\int_0^R G'(t) P_1(t)\, dt = \sum_{r=0}^{R-1} \int_0^1 G'(r+u)(u - \tfrac{1}{2})\, du$$

$$= \tfrac{1}{2}G(0) + \sum_{r=1}^{R-1} G(r) + \tfrac{1}{2}G(R) - \int_0^R G(t)\, dt. \qquad (2.1)$$

By the foregoing lemma we see that, as $R \to \infty$, the left-hand side of (2.1) tends to

$$\int_0^\infty G'(t)P_1(t)\,dt = \sum_{k=1}^{2K}(-1)^k G^{(k)}(0)P_{k+1}(0) + \int_0^\infty G^{(2K+1)}(t)P_{2K+1}(t)\,dt$$
$$= \sum_{s=1}^{K-1}\frac{(-1)^s B_s R_{2s+j}}{2s} + O(x^{2K+j}).$$

Again
$$\sum_{r=1}^{R-1} G(r) - \int_0^R G(t)\,dt = D_1 + D_2 + D_3,$$

where

$$D_1 = \sum_{r=1}^{R-1}\left\{G(r) + \frac{R_j}{r}\right\}, \qquad D_2 = R_j\left\{\int_0^R\left(\frac{1}{t} - \frac{1}{e^t-1}\right)dt - \sum_{r=1}^{R-1}\frac{1}{t}\right\},$$
$$D_3 = -\int_0^R\left\{G(t) + \frac{R_j}{t} - \frac{R_j}{e^t-1}\right\}dt.$$

Now, as $R \to \infty$,

$$D_1 = \sum_{r=1}^{R-1} L\omega(r) - \sum_{r=1}^{R-1} S_1(r) \to LF_j - \sum_{k=0}^{j-1}\zeta(j+1-k)R_k,$$
$$D_2 = R_j\left\{\log\left(\frac{R}{1-e^{-R}}\right) - \sum_{r=1}^{R-1}\frac{1}{r}\right\} \to -R_j\gamma,$$

where γ is Euler's constant, and

$$D_3 \to -\int_0^\infty\left\{L\omega(t) - S_1(t) - \frac{R_j}{e^t-1}\right\}dt = R_j\gamma - H.$$

Finally $G(0) = R_{j+1}$. Hence, if we let $R \to \infty$ in (2.1), we have Theorem 1.

3. A more exact result

Theorem 1 gives an asymptotic expansion of F_j for small $x_1, \ldots, x_j$. If $j = 1$, we find with a little calculation that

$$d = \zeta'(0) = -\tfrac{1}{2}\log(2\pi);$$

again

$$R_0 = 1, \qquad R_1 = -\tfrac{1}{2}x_1, \qquad R_2 = \tfrac{1}{12}x_1^2, \qquad R_{2s+1} = 0 \quad (s \geqslant 1).$$

Hence, by Theorem 1,

$$F_1 = \frac{\zeta(2)}{x_1} + \tfrac{1}{2}\log x_1 - \tfrac{1}{2}\log(2\pi) - \frac{x_1}{24} + O(x^{2K})$$

for every K. In fact, by (2), the error term is $O(e^{-4\pi^2/x})$. For $j > 1$,

nothing similar is true for F_j. But the logarithms of the generating functions are linear combinations of the F_j [compare the cases $s = 1, 3, 4$ of (9)] and we may look for a linear combination which will make the coefficients zero in what corresponds to the second sum of Theorem 1.

Let us consider the series

$$\bar{F}_j = \sum_{r=1}^{\infty} \left\{ \rho(r) - \frac{1}{2^j r} \right\},$$

where
$$\rho(t) = \frac{1}{t} \prod_{l=1}^{j} \left(\frac{1}{e^{x_l t} - 1} + \frac{1}{2} \right) = \frac{1}{2^j t} \prod_{l=1}^{j} \left(\frac{e^{x_l t} + 1}{e^{x_l t} - 1} \right),$$

so that
$$\bar{F}_j = F_j + \tfrac{1}{2} \sum F_{j-1} + \ldots + 2^{1-j} \sum F_1(x_l).$$

We have then, for $|t| < 2\pi/|x|$,

$$L t^{j+1} \rho(t) = \prod_{l=1}^{j} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_k x_l^{2k} t^{2k}}{(2k)!} \right\} = \sum_{k=0}^{\infty} Q_k t^k,$$

where $Q_k = 0$ for odd k and, for even k, is a homogeneous polynomial of degree $\frac{1}{2}k$ in $x_1^2, \ldots, x_j^2$. By the same method that we used to prove Theorem 1, we can prove the theorem:

THEOREM 2.

$$L\bar{F}_j = \sum_{k=0}^{j-1} \zeta(j+1-k) Q_k + \left(\frac{L}{j 2^j} - \frac{Q_j}{j} \right) \log L + L\bar{d}_j - \tfrac{1}{2} Q_{j+1} + \\ + \sum_{s=1}^{K-1} \frac{(-1)^s B_s Q_{2s+j}}{2s} + O(x^{2K+j}), \quad (3.1)$$

where $\bar{d}_j$ depends only on the ratios of $x_1, \ldots, x_j$.

If j is odd, all the terms in the second sum on the right-hand side of (3.1) vanish, and so, just as for F_1, we can endeavour to improve the error term. In fact we can obtain an exact transformation. For simplicity, we confine ourselves to the case when $j = 3$ and x_1, x_2, x_3 are real and positive.

The function
$$\lambda(t) = \frac{2\pi i \rho(t)}{1 - e^{-2\pi i t}}$$

has its poles in the top half of the complex t-plane at those points at which one or more of $x_1 t$, $x_2 t$, $x_3 t$ is an integral multiple of $2\pi i$. If the ratios of x_1, x_2, x_3 are all rational and n_1, n_2, n_3 are positive integers such that

$$\frac{n_1}{x_1} = \frac{n_2}{x_2} = \frac{n_3}{x_3} = N \quad \text{(say)},$$

the function $\lambda(t)$ has a triple pole at $t = 2\pi iN$. The residue is

$$\frac{1}{Nx_1x_2x_3(e^{4\pi^2N}-1)}\left\{\frac{1}{N(1-e^{-4\pi^2N})}+\frac{1}{4\pi^2N^2}-\frac{x_1^2+x_2^2+x_3^2}{12}+\frac{2\pi^2(1+e^{-4\pi^2N})}{(1-e^{-4\pi^2N})^2}\right\}.$$

If
$$\frac{n_1}{x_1}=\frac{n_2}{x_2}=N \quad \text{(say)},$$

but x_3N is not an integer, the function $\lambda(t)$ has a double pole at $t = 2\pi iN$ and the residue is

$$-\frac{\cot(\pi Nx_3)}{Nx_1x_2(e^{4\pi^2N}-1)}\left(\frac{\pi}{1-e^{-4\pi^2N}}+\frac{1}{4\pi N}+\frac{x_3}{2\sin(2\pi Nx_3)}\right).$$

If $n_1/x_1 = N$, but neither x_2N nor x_3N is an integer, the function $\lambda(t)$ has a simple pole at $t = 2\pi iN$, and the residue is

$$\frac{\cot(N\pi x_2)\cot(N\pi x_3)}{4Nx_1(e^{4\pi^2N}-1)}.$$

Let ϵ_2 be any fixed positive number such that $0 < \epsilon_2 < 1$; let $x = \max(x_1, x_2, x_3)$, $x' = \min(x_1, x_2, x_3)$ and $\eta = \epsilon_2 x'/x$. We can find an increasing sequence of positive real numbers V_n such that

$$\left|V_n-\frac{2\pi n_l}{x_l}\right| \geqslant \frac{\eta}{2\pi x_l} \tag{3.2}$$

for $l = 1, 2, 3$ and every non-negative integer n_l and that $V_n \to \infty$ as $n \to \infty$. We write T_n for the sum of the residues of $\lambda(t)$ at its poles at $t = iv$ for which $0 < v < V_n$. We also write $T = \lim_{n\to\infty} T_n$ if this limit exists.

THEOREM 3. *If x_1, x_2, x_3 are all real and positive, then the limit T exists and*

$$\overline{F}_3 = \frac{\zeta(4)}{L}+\frac{Q_2\zeta(2)}{L}+\tfrac{1}{24}\log L+\bar{d}_3-\frac{Q_4}{2L}-T, \tag{3.3}$$

where

$$L = x_1x_2x_3, \qquad 12Q_2 = x_1^2+x_2^2+x_3^2,$$

$$720Q_4 = 5x_2^2x_3^2+5x_3^2x_1^2+5x_1^2x_2^2-x_1^4-x_2^4-x_3^4,$$

$$\bar{d}_3 = \int_0^\infty\left\{\rho(t)-\frac{1}{Lt^4}-\frac{Q_2}{Lt^2}-\frac{1}{8t}+\frac{1}{8(e^t-1)}\right\}dt-\frac{\gamma}{8}-\frac{\log L}{24}$$

and $\bar{d}_3$ depends only on the ratios of x_1, x_2, x_3.

Also
$$T = O\left(\frac{1}{\eta^3}\exp\left\{-\frac{(4\pi^2-\eta)}{x}\right\}\right). \tag{3.4}$$

4. Proof of Theorem 3

Here $j = 3$ and we write

$$\bar{G}(t) = \rho(t) - \frac{1}{Lt^4} - \frac{Q_2}{Lt^2},$$

so that $\bar{G}(t)$ is even and regular at $t = 0$. We take R and n any positive integers and $V = V_n$. The contour $\mathscr{C}_1$ is the straight line from $t = -R-\frac{1}{2}$ to $t = R+\frac{1}{2}$ indented above the points $t = -R, -R+1, \ldots, 0, 1, \ldots, R$; $\mathscr{C}_2$ is the line from $t = R+\frac{1}{2}$ to $t = R+\frac{1}{2}+iV$; $\mathscr{C}_3$ the line from $t = R+\frac{1}{2}+iV$ to $t = -R-\frac{1}{2}+iV$; and $\mathscr{C}_4$ the line from $t = -R-\frac{1}{2}+iV$ to $t = -R-\frac{1}{2}$. We use $\mathscr{C}$ for the complete contour $\mathscr{C}_1+\mathscr{C}_2+\mathscr{C}_3+\mathscr{C}_4$. The function $\rho(t)-\bar{G}(t)$ is regular within $\mathscr{C}$. Hence

$$T_n = \frac{1}{2\pi i}\int_{\mathscr{C}} \lambda(t)\,dt = \int_{\mathscr{C}} \frac{\bar{G}(t)\,dt}{1-e^{-2\pi i t}} = \int_{\mathscr{C}_1} + \ldots + \int_{\mathscr{C}_4} = E_1 + \ldots + E_4 \quad (4.1)$$

(say).

If $\overline{\mathscr{C}_1}$ is the contour formed by reflecting $\mathscr{C}_1$ in the real axis, we have

$$2E_1 = \int_{\mathscr{C}_1} \frac{\bar{G}(t)\,dt}{1-e^{-2\pi i t}} - \int_{\overline{\mathscr{C}_1}} \frac{\bar{G}(t)\,dt}{e^{2\pi i t}-1} = \int_{-R-\frac{1}{2}}^{R+\frac{1}{2}} \bar{G}(t)\,dt - \sum_{r=-R}^{R} \bar{G}(r),$$

if we let the radii of the indentations tend to 0. Hence

$$\begin{aligned} E_1 &= \int_0^{R+\frac{1}{2}} \bar{G}(t)\,dt - \tfrac{1}{2}\bar{G}(0) - \sum_{r=1}^{R} \bar{G}(r) \\ &= \int_0^{R+\frac{1}{2}} \left\{\bar{G}(t) - \frac{1}{8t} + \frac{1}{8(e^t-1)}\right\} dt - \sum_{r=1}^{R} \left\{\bar{G}(r) - \frac{1}{8r}\right\} - \frac{Q_4}{2L} + \\ &\qquad + \tfrac{1}{8}\log\left(\frac{R+\frac{1}{2}}{1-e^{-R-\frac{1}{2}}}\right) - \frac{1}{8}\sum_{r=1}^{R}\frac{1}{r} \end{aligned}$$

$$\to \bar{d}_3 - \bar{F}_3 + \frac{\zeta(4)}{L} + \frac{Q_2\zeta(2)}{L} + \frac{\log L}{24} - \frac{Q_4}{2L} \quad \text{as } R \to \infty. \quad (4.2)$$

On $\mathscr{C}_2$, $\qquad t = R+\frac{1}{2}+iv, \qquad \rho(t) \sim 2^{-j}t^{-1}$

for large R. Hence

$$|\bar{G}(t)| = \left|\rho(t) - \frac{1}{Lt^4} - \frac{Q_2}{Lt^2}\right| < \frac{C}{R} + \frac{1}{LR^4} + \frac{Q_2}{LR^2}$$

and

$$|E_2| \leqslant \left(\frac{C}{R} + \frac{1}{LR^4} + \frac{Q_2}{LR^2}\right)\int_0^{\infty} \frac{dv}{1+e^{2\pi v}} \to 0$$

uniformly in V as $R \to \infty$. A similar result holds for E_4. Hence, using (4.1) and (4.2), we have

$$\bar{F}_3 = \frac{\zeta(4)}{L} + \frac{Q_2\zeta(2)}{L} + \frac{\log L}{24} + \bar{d}_3 - \frac{Q_4}{2L} - T_n + E_5, \quad (4.3)$$

where
$$E_5 = \lim_{R\to\infty} E_3 = E_6+E_7+E_8 \tag{4.4}$$
and
$$E_6 = -\int_{-\infty+iV}^{iV} \frac{\rho(t)\,dt}{1-e^{-2\pi it}}, \qquad E_7 = -\int_{iV}^{\infty+iV} \frac{\rho(t)\,dt}{1-e^{-2\pi it}},$$
$$E_8 = \int_{-\infty+iV}^{\infty+iV} \frac{(1+Q_2t^2)\,dt}{Lt^4(1-e^{-2\pi it})}.$$

The integrand in E_8 has no poles above the line $\operatorname{im} t = V$ and a simple application of Cauchy's theorem shows that $E_8 = 0$.

When $t = u+iV$, we have
$$|e^{x_1t}-1|^2 = |e^{x_1u+ix_1V}-1|^2 = e^{2x_1u}-2e^{x_1u}\cos x_1V+1$$
$$= (e^{x_1u}-1)^2+4e^{x_1u}\sin^2(\tfrac{1}{2}x_1V) \geqslant C\eta^2e^{x_1u},$$
by (3.2). A similar result holds for $|e^{x_3t}-1|^2$ and $|e^{x_5t}-1|^2$. Hence
$$\left|\rho(t)-\frac{1}{8t}\right| = \left|\frac{1}{8t}\left\{\left(1+\frac{2}{e^{x_1t}-1}\right)\left(1+\frac{2}{e^{x_3t}-1}\right)\left(1+\frac{2}{e^{x_5t}-1}\right)-1\right\}\right|$$
$$\leqslant \frac{C}{V}(\eta^{-2}e^{-\frac{1}{2}x'u}+\eta^{-3}e^{-\frac{1}{2}xu}).$$

Now
$$-E_7 = \int_{iV}^{\infty+iV} \frac{dt}{8t(1-e^{-2\pi it})} + \int_{iV}^{\infty+iV} \frac{\rho(t)-(8t)^{-1}}{1-e^{-2\pi it}}\,dt = E_9+E_{10}$$
(say), and we have
$$|E_{10}| \leqslant \frac{Ce^{-2\pi V}}{V\eta^3}\int_0^\infty (\eta e^{-\frac{1}{2}x'u}+e^{-\frac{1}{2}xu})\,du \leqslant \frac{Ce^{-2\pi V}}{Vx\eta^3}.$$

Again,
$$8E_9 = \int_{iV}^{\infty+iV} \frac{dt}{t(1-e^{-2\pi it})} = \frac{\log(1-e^{-2\pi V})}{2\pi V} + \int_{iV}^{\infty+iV} \frac{\log(1-e^{2\pi it})\,dt}{2\pi it^2},$$
and so
$$|E_9| < \frac{Ce^{-2\pi V}}{V} + Ce^{-2\pi V}\int_0^\infty \frac{du}{u^2+V^2} < \frac{Ce^{-2\pi V}}{V}.$$
Hence $|E_7| < Ce^{-2\pi V}/(Vx\eta^3)$.

Similar arguments apply to E_6, which is in any case the conjugate of E_7. Hence we have
$$|E_5| \leqslant |E_6|+|E_7| \leqslant Ce^{-2\pi V}/(Vx\eta^3). \tag{4.5}$$
If $n \to \infty$, then $V \to \infty$, $E_5 \to 0$, and, from (4.3), T_n tends to a limit T, and (3.3) follows.

It also follows that, for any $n \geqslant 1$,
$$T_n - E_5 = T.$$

If we take $$V_1 = \frac{1}{x}\left(2\pi - \frac{\eta}{2\pi}\right),$$

then (3.2) is satisfied, $T_1 = 0$ and $T = -E_5$. Hence (3.4) follows from (4.5).

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SOME GENERALIZATIONS OF TAUBER'S SECOND THEOREM

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1. In some recent papers (**1**, **2**, **3**) I have discussed an extension of absolute Cesàro summability to index k. These papers are concerned primarily with those parts of the subject which are of interest in function-theory. In this paper I consider the summability aspect and show how the use of an index k enables us to span the gap between ordinary convergence and summability on the one hand and absolute convergence and absolute summability on the other.

We begin by recalling the familiar generalization to order α of Tauber's well-known 'second theorem'.† Let $\sum_0^\infty a_n$ be a series of constant terms, and let σ_n^α and τ_n^α be the nth Cesàro means of order α of the series $\sum a_n$ and the sequence na_n respectively. Let also

$$\phi(x) = \sum_0^\infty a_n x^n,$$

this series being supposed convergent for $|x| < 1$. Then we have

THEOREM A. *If*
$$\phi(x) - s = o(1) \tag{1.1}$$
as $x \to 1-$, *and if, for some* $\alpha > 0$,
$$\tau_n^\alpha = o(1) \tag{1.2}$$
as $n \to \infty$, *then*
$$\sigma_n^{\alpha-1} - s = o(1). \tag{1.3}$$

We note also the inequality form of Theorem A:‡

THEOREM B. *If* $\alpha > 0$, *then for any* s

$$\sup_m |\sigma_m^{\alpha-1} - s| \leqslant A(\alpha) \sup_m |\tau_m^\alpha| + \sup_{0 \leqslant x < 1} |\phi(x) - s|.$$

I use $A(b, c, \ldots)$ to denote a positive constant depending only on b, c,..., not necessarily the same on any two occurrences. Inequalities of the form

$$L \leqslant A(b, c, \ldots) R$$

are to be interpreted as meaning 'if the expression R is finite, then the expression L is finite and satisfies the inequality'.

† See, for example, Kogbetliantz (**7**).

‡ A proof of Theorem B is given in (**1**). The s is actually irrelevant here, but it becomes relevant in the extension of Theorem B to index k.

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We observe now that the relations (1.3) and (1.2) are the limiting forms as $k \to \infty$ of the relations

$$\left\{m^p \sum_{n=m}^{\infty} \frac{|\sigma_n^{\alpha-1}-s|^k}{(n+1)^\gamma}\right\}^{1/k} = o(1) \tag{1.4}$$

and

$$\left\{m^p \sum_{n=m}^{\infty} \frac{|\tau_n^{\alpha}|^k}{n^\gamma}\right\}^{1/k} = o(1), \tag{1.5}$$

respectively, where p and γ are some fixed numbers.† Similarly (1.1) is the limiting form of the relation

$$\left\{(1-x)^{-p} \int_x^1 \frac{|\phi(y)-s|^k}{(1-y)^{2-\gamma}}\,dy\right\}^{1/k} = o(1). \tag{1.6}$$

The natural extension of Theorem A to index k is therefore that for suitable p and γ the conditions (1.5) and (1.6) together imply (1.4). The most interesting case of the result appears to be that in which $p = \gamma-1 \geqslant 0$, and to avoid a multiplicity of parameters we confine ourselves to this case. Here we have

THEOREM 1. *If $k \geqslant 1$, $\alpha > 0$, and $p = \gamma-1 \geqslant 0$, then* (1.5) *and* (1.6) *together imply* (1.4).

When $p = 0$, (1.4) merely asserts the convergence of the series

$$\sum \frac{|\sigma_n^{\alpha-1}-s|^k}{n+1},$$

and similarly in the case of (1.5) and (1.6). We can therefore strengthen the case $p = 0$ of Theorem 1 by stating it in the following inequality form.

THEOREM 2. *Let $k \geqslant 1$ and $\alpha > 0$. Then for any s*

$$\left\{\sum_0^{\infty} \frac{|\sigma_n^{\alpha-1}-s|^k}{n+1}\right\}^{1/k} \leqslant A(k,\alpha)\left\{\sum_1^{\infty} \frac{|\tau_n^{\alpha}|^k}{n}\right\}^{1/k} + \left\{\int_0^1 \frac{|\phi(x)-s|^k}{1-x}\,dx\right\}^{1/k}.$$

This result is simply a direct extension of Theorem B to index k, but, whereas Theorem B does not imply Theorem A, Theorem 2 does imply the appropriate case of Theorem 1.

Theorem 2 is concerned with the absolute summability of the series

$$\sum_0^{\infty} \frac{(\sigma_n^0-s)}{n+1}$$

† See, for example, Littlewood (8) 16.

rather than with that of the series $\sum a_n$. We recall that a series $\sum a_n$ is summable $|C,\alpha|_k$, where $k \geqslant 1$ and $\alpha > -1$, if the series

$$\sum_1^\infty n^{k-1}|\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k \quad \left(= \sum_1^\infty n^{-1}|\tau_n^\alpha|^k\right)$$

converges, and that it is summable $|A|_k$ if the integral

$$\int_0^1 (1-x)^{k-1}|\phi'(x)|^k\,dx$$

converges. Summability $|C,\alpha|_1$ and summability $|A|_1$ are, of course, identical with summability $|C,\alpha|$ and summability $|A|$, respectively.

To obtain a theorem concerning the absolute summability of $\sum a_n$, we apply Theorem 2, with α replaced by $\alpha+1$, to the series

$$\sum \{na_n - (n-1)a_{n-1}\}.$$

Observing that τ_n^α is the nth (C,α)-mean of this series, and that

$$\sum_1^\infty \{na_n - (n-1)a_{n-1}\}x^n = (1-x)x\phi'(x),$$

we obtain immediately the following theorem, of which the case $k = 1$ is an extension of a well-known Tauberian theorem due to Hyslop (**5**).†

THEOREM 3. *Let $k \geqslant 1$ and $\alpha > -1$. If $\sum a_n$ is summable $|A|_k$, and if also $\sum \Delta(na_n)$‡ is summable $|C,\alpha+1|_k$, then $\sum a_n$ is summable $|C,\alpha|_k$.*

The case $p > 0$ of Theorem 1 has applications to strong summability. A series $\sum a_n$ is said to be 'summable $[C,\alpha]_k$ to the sum s', where $k \geqslant 1$ and $\alpha > 0$, if

$$\sum_0^m |\sigma_n^{\alpha-1} - s|^k = o(m) \tag{1.7}$$

as $m \to \infty$. Summability $[C,1]_k$ is, of course, ordinary strong summability. The property of $\phi(x)$ corresponding to summability $[C,\alpha]_k$ as summability (A) corresponds to summability (C) is that

$$\int_0^R \frac{|\phi(x)-s|^k}{(1-x)^2}\,dx = o\left(\frac{1}{1-R}\right) \tag{1.8}$$

as $R \to 1-$, and we shall say that a series which possesses the property (1.8) is 'summable $[A]_k$ to the sum s'.§

It follows easily by partial summation and integration by parts that, when $p = \gamma - 1 > 0$, (1.4) and (1.6) are equivalent to (1.7) and (1.8)

† Hyslop deals only with the case $\alpha \geqslant 0$.

‡ As usual, $\Delta u_n = u_n - u_{n+1}$.

§ For a fuller discussion of the relation between summability $[C,\alpha]_k$ and summability $[A]_k$, see Flett (**4**).

respectively, while (1.5) is equivalent to the relation

$$\sum_1^m |\tau_n^\alpha|^k = o(m). \tag{1.9}$$

We may therefore restate the case $p > 0$ of Theorem 1 in the equivalent form:

THEOREM 4. *Let $k \geqslant 1$ and $\alpha > 0$. If $\sum a_n$ is summable $[A]_k$ to the sum s, and if (1.9) is satisfied, then $\sum a_n$ is summable $[C, \alpha]_k$ to the sum s.*

Since (1.8) is satisfied whenever $\sum a_n$ is summable (A) to the sum s, Theorem 4 is a generalization of a Tauberian theorem of the author's (**3**).

There is also a boundedness result corresponding to Theorem 4, namely:†

THEOREM 5. *Let $k \geqslant 1$ and $\alpha > 0$. Then*

$$\sup_m \left\{\frac{1}{m+1}\sum_0^m |\sigma_n^{\alpha-1}|^k\right\}^{1/k} \leqslant A(k,\alpha)\sup_m \left\{\frac{1}{m+1}\sum_1^m |\tau_n^\alpha|^k\right\}^{1/k} + \\ + A(k) \sup_{0\leqslant R<1} \left\{(1-R)\int_0^R \frac{|\phi(x)|^k}{(1-x)^2}dx\right\}^{1/k}.$$

The proof of Theorem 2 is given in §§ 2–4. We prove Theorem 4 in preference to the case $p > 0$ of Theorem 1,‡ and the proofs of this and Theorem 5 are given in § 5.

2. In the proof of Theorem 2 we require the following lemmas:

LEMMA 1. *If $k \geqslant 1$, $a_n = a_n(x)$, and $b_n = b_n(x)$, then*

$$\left\{\int (\sum |a_n+b_n|^k)\,dx\right\}^{1/k} \leqslant \left\{\int (\sum |a_n|^k)\,dx\right\}^{1/k} + \left\{\int (\sum |b_n|^k)\,dx\right\}^{1/k}.$$

This is one of the numerous forms of Minkowski's inequality.

LEMMA 2. *If $\beta > 0$ and $\gamma \geqslant 1$, then*

$$A(\beta)\gamma^{-\beta} \leqslant \int_0^1 (1-x)^{\beta-1}x^\gamma\,dx \leqslant \Gamma(\beta)\gamma^{-\beta}.$$

This lemma is given by Mulholland (**9**).

LEMMA 3. *Let*

$$E_n^\alpha = \binom{\alpha+n}{n} = \frac{(\alpha+1)(\ldots)(\alpha+n)}{n!} \quad (n \geqslant 1), \qquad E_0^\alpha = 1.$$

Then if $\alpha \geqslant 0$, $0 \leqslant x < 1$, and $s \geqslant 1$,

$$\sum_{n=s}^\infty E_n^\alpha x^n \leqslant 2^\alpha x^s\{sE_s^\alpha + (1-x)^{-\alpha-1}\}.$$

† In this result s is clearly irrelevant.

‡ This enables us to avoid the introduction of the parameter p.

We have $$\sum_{n=0}^{\infty} E_n^{\alpha} x^n = (1-x)^{-\alpha-1}.$$

Hence, by the well-known integral formula for the remainder after $s-1$ terms in a Taylor's series,

$$\sum_{n=s}^{\infty} E_n^{\alpha} x^n = \frac{x^s(\alpha+1)(\ldots)(\alpha+s)}{(s-1)!} \int_0^1 (1-t)^{s-1}(1-tx)^{-\alpha-s-1}\,dt$$

$$= sx^s E_s^{\alpha} \int_0^1 (1-t)^{s-1}(1-tx)^{-\alpha-s-1}\,dt. \tag{2.1}$$

Writing $v = (1-x)t/(1-t)$, we see easily that the integral in (2.1) is equal to

$$(1-x)^{-\alpha-1} \int_0^{\infty} \frac{\{v+(1-x)\}^{\alpha}}{(v+1)^{s+\alpha+1}}\,dv.$$

Further,

$$\int_0^{1-x} \frac{\{v+(1-x)\}^{\alpha}}{(v+1)^{s+\alpha+1}}\,dv \leqslant 2^{\alpha}(1-x)^{\alpha} \int_0^{1-x} dv = 2^{\alpha}(1-x)^{\alpha+1},$$

and

$$\int_{1-x}^{\infty} \frac{\{v+(1-x)\}^{\alpha}}{(v+1)^{s+\alpha+1}}\,dv \leqslant 2^{\alpha} \int_0^{\infty} \frac{v^{\alpha}}{(v+1)^{s+\alpha+1}}\,dv = \frac{2^{\alpha}\Gamma(\alpha+1)\Gamma(s)}{\Gamma(s+\alpha+1)} = \frac{2^{\alpha}}{sE_s^{\alpha}},$$

and this completes the proof.

3. We require also the well-known identity, valid for all α,

$$\phi(x) = (1-x)^{\alpha+1} \sum_0^{\infty} E_n^{\alpha} \sigma_n^{\alpha} x^n \quad (|x| < 1). \tag{3.1}$$

Here E_n^{α} is defined as in Lemma 3. We have also†

$$\tau_n^{\alpha} = n(\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}), \tag{3.2}$$

and, for $\alpha > 0$,

$$\tau_n^{\alpha} = -\alpha(\sigma_n^{\alpha} - \sigma_n^{\alpha-1}). \tag{3.3}$$

4. Consider now the proof of Theorem 2. We may evidently suppose that $s = 0$. Further, by (3.3),

$$|\sigma_n^{\alpha-1}| \leqslant |\sigma_n^{\alpha}| + \alpha^{-1}|\tau_n^{\alpha}|,$$

so that, by Minkowski's inequality,

$$\left\{\sum_0^{\infty} \frac{|\sigma_n^{\alpha-1}|^k}{n+1}\right\}^{1/k} \leqslant \left\{\sum_0^{\infty} \frac{|\sigma_n^{\alpha}|^k}{n+1}\right\}^{1/k} + \alpha^{-1}\left\{\sum_1^{\infty} \frac{|\tau_n^{\alpha}|^k}{n+1}\right\}^{1/k}.$$

† Kogbetliantz (6).

It is therefore enough to prove that

$$\left\{\sum_0^\infty \frac{|\sigma_n^\alpha|^k}{n+1}\right\}^{1/k} \leqslant A(k,\alpha)\left\{\sum_1^\infty \frac{|\tau_n^\alpha|^k}{n}\right\}^{1/k} + \left\{\int_0^1 \frac{|\phi(x)|^k}{1-x}\,dx\right\}^{1/k}$$

when $\alpha > 0$.

We observe next that†

$$\sum_0^\infty \frac{|\sigma_n^\alpha|^k}{n+1} = \int_0^1 \left(\sum_0^\infty |\sigma_n^\alpha|^k x^n\right) dx.$$

Hence, by Lemma 1,

$$\left\{\sum_0^\infty \frac{|\sigma_n^\alpha|^k}{n+1}\right\}^{1/k} \leqslant \left\{\int_0^1 \left(\sum_0^\infty |\sigma_n^\alpha - \phi(x)|^k x^n\right) dx\right\}^{1/k} + \left\{\int_0^1 \left(\sum_0^\infty |\phi(x)|^k x^n\right) dx\right\}^{1/k}$$

$$= \left\{\int_0^1 \left(\sum_0^\infty |\sigma_n^\alpha - \phi(x)|^k x^n\right) dx\right\}^{1/k} + \left\{\int_0^1 \frac{|\phi(x)|^k}{1-x}\,dx\right\}^{1/k}.$$

It is therefore enough to prove that, if $\alpha > 0$ and $k \geqslant 1$, then

$$\int_0^1 \left(\sum_0^\infty |\sigma_n^\alpha - \phi(x)|^k x^n\right) dx \leqslant A(k,\alpha) \sum_1^\infty n^{-1} |\tau_n^\alpha|^k. \tag{4.1}$$

We now suppose that $0 \leqslant x < 1$. We also write B for a constant depending on either or both of k and α. By (3.1), we have

$$\sigma_n^\alpha - \phi(x) = (1-x)^{\alpha+1} \sum_0^\infty E_\nu^\alpha (\sigma_n^\alpha - \sigma_\nu^\alpha) x^\nu,$$

so that

$$|\sigma_n^\alpha - \phi(x)|^k \leqslant (1-x)^{k(\alpha+1)} \left(\sum_0^\infty E_\nu^\alpha |\sigma_n^\alpha - \sigma_\nu^\alpha| x^\nu\right)^k$$

$$\leqslant (1-x)^{k(\alpha+1)} \left(\sum_0^\infty E_\nu^\alpha |\sigma_n^\alpha - \sigma_\nu^\alpha|^k x^\nu\right)\left(\sum_0^\infty E_\nu^\alpha x^\nu\right)^{k-1}$$

$$= (1-x)^{\alpha+1} \sum_0^\infty E_\nu^\alpha |\sigma_n^\alpha - \sigma_\nu^\alpha|^k x^\nu.$$

If then

$$S_n = (1-x)^{\alpha+1} \sum_0^{n-1} E_\nu^\alpha |\sigma_n^\alpha - \sigma_\nu^\alpha|^k x^\nu,$$

and

$$T_n = (1-x)^{\alpha+1} \sum_{n+1}^\infty E_\nu^\alpha |\sigma_n^\alpha - \sigma_\nu^\alpha|^k x^\nu, \tag{4.2}$$

we have

$$|\sigma_n^\alpha - \phi(x)|^k \leqslant S_n + T_n. \tag{4.3}$$

† The term-by-term integration is legitimate, since everything is positive. Similar points occur elsewhere, and I shall in future take such justification for granted.

Consider now S_n and suppose first that $k > 1$. If $\nu < n$, then, by (3.2),†

$$|\sigma_n^\alpha - \sigma_\nu^\alpha|^k = \Big| \sum_{\nu+1}^{n} s^{-1}\tau_s^\alpha \Big|^k \leqslant \Big(\sum_{\nu+1}^{n} |\tau_s^\alpha|^k \Big) \Big(\sum_{\nu+1}^{n} s^{-k'} \Big)^{k/k'},$$

so that

$$|\sigma_n^\alpha - \sigma_\nu^\alpha|^k \leqslant B(\nu+1)^{-1} \sum_{\nu+1}^{n} |\tau_s^\alpha|^k. \tag{4.4}$$

On the other hand, if $k = 1$, then (4.4) is an immediate consequence of (3.2), so that (4.4) holds for all $k \geqslant 1$. Since $\alpha > 0$, we have therefore

$$\begin{aligned}(1-x)^{-\alpha-1} S_n &\leqslant B \sum_{\nu=0}^{n-1} E_\nu^{\alpha-1} x^\nu \sum_{s=\nu+1}^{n} |\tau_s^\alpha|^k = B \sum_{s=1}^{n} |\tau_s^\alpha|^k \sum_{\nu=0}^{s-1} E_\nu^{\alpha-1} x^\nu \\ &\leqslant B \sum_{s=1}^{n} |\tau_s^\alpha|^k \sum_{\nu=0}^{\infty} E_\nu^{\alpha-1} x^\nu = B(1-x)^{-\alpha} \sum_{s=1}^{n} |\tau_s^\alpha|^k. \end{aligned} \tag{4.5}$$

Hence also

$$\begin{aligned}\sum_{n=1}^{\infty} S_n x^n &\leqslant B(1-x) \sum_{n=1}^{\infty} x^n \sum_{s=1}^{n} |\tau_s^\alpha|^k = B(1-x) \sum_{s=1}^{\infty} |\tau_s^\alpha|^k \sum_{n=s}^{\infty} x^n \\ &\leqslant B \sum_{s=1}^{\infty} |\tau_s^\alpha|^k x^s, \end{aligned}$$

so that

$$\int_0^1 \Big(\sum_{n=1}^{\infty} S_n x^n \Big) dx \leqslant B \sum_{s=1}^{\infty} (s+1)^{-1} |\tau_s^\alpha|^k \leqslant B \sum_{s=1}^{\infty} s^{-1} |\tau_s^\alpha|^k. \tag{4.6}$$

Consider next T_n. Let μ be a number, depending on k, such that $0 < k\mu < 1$, e.g. $\mu = (2k)^{-1}$. Then, if $k > 1$, we have $k'(1-\mu) > 1$. If now $\nu > n$, we obtain

$$|\sigma_n^\alpha - \sigma_\nu^\alpha|^k = \Big| \sum_{n+1}^{\nu} s^{-1}\tau_s^\alpha \Big|^k \leqslant \Big(\sum_{n+1}^{\nu} s^{-k\mu} |\tau_s^\alpha|^k \Big) \Big(\sum_{n+1}^{\nu} s^{-k'(1-\mu)} \Big)^{k/k'}$$

for $k > 1$, so that

$$|\sigma_n^\alpha - \sigma_\nu^\alpha|^k \leqslant B(n+1)^{k\mu-1} \sum_{n+1}^{\nu} s^{-k\mu} |\tau_s^\alpha|^k. \tag{4.7}$$

On the other hand, if $k = 1$, then (4.7) is an immediate consequence of (3.2), so that (4.7) holds for all $k \geqslant 1$.

By (4.2), (4.7), and Lemma 3, we have now

$$\begin{aligned} T_n &\leqslant B(1-x)^{\alpha+1}(n+1)^{k\mu-1} \sum_{\nu=n+1}^{\infty} E_\nu^\alpha x^\nu \sum_{s=n+1}^{\nu} s^{-k\mu} |\tau_s^\alpha|^k \\ &= B(1-x)^{\alpha+1}(n+1)^{k\mu-1} \sum_{s=n+1}^{\infty} s^{-k\mu} |\tau_s^\alpha|^k \sum_{\nu=s}^{\infty} E_\nu^\alpha x^\nu \\ &\leqslant B(U_n + V_n), \end{aligned} \tag{4.8}$$

where

$$U_n = (1-x)^{\alpha+1}(n+1)^{k\mu-1} \sum_{s=n+1}^{\infty} s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s \tag{4.9}$$

and

$$V_n = (n+1)^{k\mu-1} \sum_{s=n+1}^{\infty} s^{-k\mu} |\tau_s^\alpha|^k x^s. \tag{4.10}$$

† As usual, $k' = k/(k-1)$.

Further,

$$\sum_0^\infty U_n x^n = (1-x)^{\alpha+1} \sum_{n=0}^\infty (n+1)^{k\mu-1} x^n \sum_{s=n+1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s$$

$$= (1-x)^{\alpha+1} \sum_{s=1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s \sum_{n=0}^{s-1} (n+1)^{k\mu-1} x^n$$

$$\leqslant B(1-x)^{\alpha-k\mu+1} \sum_{s=1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s,$$

so that, by Lemma 2,

$$\int_0^1 \Big(\sum_0^\infty U_n x^n \Big) dx \leqslant B \sum_{s=1}^\infty s^{-1} |\tau_s^\alpha|^k. \tag{4.11}$$

Again,

$$\sum_0^\infty V_n x^n = \sum_{n=0}^\infty (n+1)^{k\mu-1} x^n \sum_{s=n+1}^\infty s^{-k\mu} |\tau_s^\alpha|^k x^s$$

$$= \sum_{s=1}^\infty s^{-k\mu} |\tau_s^\alpha|^k x^s \sum_{n=0}^{s-1} (n+1)^{k\mu-1} x^n \leqslant B(1-x)^{-k\mu} \sum_{s=1}^\infty s^{-k\mu} |\tau_s^\alpha|^k x^s$$

(since $k\mu > 0$), whence, by Lemma 2,

$$\int_0^1 \Big(\sum_0^\infty V_n x^n \Big) dx \leqslant B \sum_{s=1}^\infty s^{-1} |\tau_s^\alpha|^k \tag{4.12}$$

(since $k\mu < 1$). Combining (4.3), (4.6), (4.8), (4.11), and (4.12) we obtain now (4.1), and this completes the proof of Theorem 2.

5. We turn next to Theorems 4 and 5, and here we require a further lemma:

LEMMA 4. *Let* $\gamma < 1$, *let* $p_n \geqslant 0$, $q_n = n^{-1} \sum_{\nu=1}^n p_\nu$ *and* $q = \sup_n q_n$. *Then*

$$\sum_{\nu=1}^n \nu^{-\gamma} p_\nu \leqslant A(\gamma) q n^{1-\gamma}.$$

If also $q_n = o(1)$ *as* $n \to \infty$, *then*

$$\sum_{\nu=1}^n \nu^{-\gamma} p_\nu = o(n^{1-\gamma}).$$

By partial summation,

$$\sum_{\nu=1}^n \nu^{-\gamma} p_\nu = \sum_{\nu=1}^{n-1} \nu q_\nu \Delta \nu^{-\gamma} + n q_n n^{-\gamma} \leqslant A(\gamma) \sum_{\nu=1}^{n-1} \nu^{-\gamma} q_\nu + n^{1-\gamma} q_n,$$

whence the result follows.

Consider now the proof of Theorem 4. As in the case of Theorem 2, we may suppose that $s = 0$. By virtue of (3.3), it is also enough to prove that, if

$$\int_0^R \frac{|\phi(x)|^k}{(1-x)^2} dx = o\left(\frac{1}{1-R}\right) \tag{5.1}$$

as $R \to 1-$, and if $$\sum_1^m |\tau_n^\alpha|^k = o(m) \tag{5.2}$$

as $m \to \infty$, then $$\sum_0^m |\sigma_n^\alpha|^k = o(m).$$

We observe next that

$$\sum_0^m |\sigma_n^\alpha|^k R^{n+1} = \int_0^R \Big(\sum_0^m (n+1)|\sigma_n^\alpha|^k x^n \Big)\, dx,$$

so that, by Lemma 1,

$$\begin{aligned} \Big\{ \sum_0^m |\sigma_n^\alpha|^k R^{n+1} \Big\}^{1/k} &\leqslant \Big\{ \int_0^R \Big(\sum_0^m (n+1)|\sigma_n^\alpha - \phi(x)|^k x^n \Big)\, dx \Big\}^{1/k} + \\ &\quad + \Big\{ \int_0^R \Big(\sum_0^m (n+1)|\phi(x)|^k x^n \Big)\, dx \Big\}^{1/k} \\ &\leqslant \Big\{ \int_0^R \Big(\sum_0^m (n+1)|\sigma_n^\alpha - \phi(x)|^k x^n \Big)\, dx \Big\}^{1/k} + \\ &\quad + \Big\{ \int_0^R \frac{|\phi(x)|^k}{(1-x)^2}\, dx \Big\}^{1/k}. \end{aligned} \tag{5.3}$$

Taking now $R = 1-1/(m+1)$ and using (5.1), we see easily that it is enough to prove that, if (5.2) is satisfied, then the first term on the right of (5.3) is $o(m^{1/k})$.

Let μ be chosen as in the proof of Theorem 2, and, as before, let us write B for a constant depending on either or both of k and α. Then from (4.3), (4.5), and (4.8) we have

$$|\sigma_n^\alpha - \phi(x)|^k \leqslant B(S_n' + U_n + V_n), \tag{5.4}$$

where U_n and V_n are defined by (4.9) and (4.10) respectively, and

$$S_n' = (1-x) \sum_{s=0}^n |\tau_s^\alpha|^k.$$

By (5.2), we have $S_n' = (1-x)o(n)$ as $n \to \infty$, where the o is independent of x, so that, as $x \to 1-$,

$$\sum_0^m (n+1) S_n' x^n \leqslant \sum_0^\infty = (1-x) \sum_0^\infty o(n^2) x^n = o\{(1-x)^{-2}\},$$

whence also

$$\int_0^R \Big(\sum_0^m (n+1) S_n' x^n \Big)\, dx = \int_0^R o\{(1-x)^{-2}\}\, dx = o\{(1-R)^{-1}\} = o(m). \tag{5.5}$$

Next, by (4.9),

$$\sum_0^m (n+1)U_n x^n = (1-x)^{\alpha+1} \sum_{n=0}^m (n+1)^{k\mu} x^n \sum_{s=n+1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s$$

$$= (1-x)^{\alpha+1} \sum_{s=1}^m s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s \sum_{n=0}^{s-1} (n+1)^{k\mu} x^n$$

$$+ (1-x)^{\alpha+1} \sum_{n=0}^m (n+1)^{k\mu} x^n \sum_{s=m+1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s$$

$$= W_m + X_m, \quad \text{say.} \tag{5.6}$$

Here $$W_m \leqslant (1-x)^{\alpha-k\mu} \sum_1^m s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s,$$

whence, by Lemma 2 and (5.2),

$$\int_0^R W_m \, dx \leqslant \int_0^1 W_m \, dx \leqslant B \sum_1^m |\tau_s^\alpha|^k = o(m). \tag{5.7}$$

Again, by partial summation and Lemmas 4 and 3, we have

$$\sum_{s=m+1}^\infty s^{\alpha-k\mu+1} |\tau_s^\alpha|^k x^s$$

$$\leqslant (1-x) \sum_{s=m+1}^\infty o(s^{\alpha-k\mu+2}) x^s \leqslant o(1)(1-x) \sum_{s=m+1}^\infty E_s^{\alpha-k\mu+2} x^s$$

$$\leqslant o(m^{\alpha-k\mu+3})(1-x)x^m + o(1)(1-x)^{-(\alpha-k\mu+2)} x^m$$

as $m \to \infty$, uniformly in $0 \leqslant x < 1$, so that

$$X_m \leqslant o(m^{\alpha-k\mu+3})(1-x)^{\alpha+2} x^m \sum_{n=0}^m (n+1)^{k\mu} x^n +$$

$$+ o(1)(1-x)^{k\mu-1} x^m \sum_{n=0}^m (n+1)^{k\mu} x^n$$

$$= o(m^{\alpha-k\mu+3})(1-x)^{\alpha-k\mu+1} x^m + o(1)(1-x)^{-2} x^m$$

$$= o(m^{\alpha-k\mu+3})(1-x)^{\alpha-k\mu+1} x^m + o(m^2) x^m$$

uniformly in $0 \leqslant x \leqslant R$. Hence also

$$\int_0^R X_m \, dx \leqslant o(m^{\alpha-k\mu+3}) \int_0^1 (1-x)^{\alpha-k\mu+1} x^m \, dx + o(m^2) \int_0^1 x^m \, dx = o(m). \tag{5.8}$$

Consider finally V_n. By (4.10),

$$\sum_0^m (n+1)V_n x^n = \sum_{n=0}^m (n+1)^{k\mu} x^n \sum_{s=n+1}^\infty s^{-k\mu} |\tau_s^\alpha|^k x^s$$

$$= \sum_{s=1}^m s^{-k\mu} |\tau_s^\alpha|^k x^s \sum_{n=0}^{s-1} (n+1)^{k\mu} x^n +$$

$$+ \sum_{n=0}^m (n+1)^{k\mu} x^n \sum_{s=m+1}^\infty s^{-k\mu} |\tau_s^\alpha|^k x^s$$

$$= Y_m + Z_m, \quad \text{say.} \tag{5.9}$$

Since $k\mu < 1$, we have, from Lemma 4,

$$Y_m \leqslant B(1-x)^{-k\mu-1} \sum_{s=1}^{m} s^{-k\mu}|\tau_s^\alpha|^k = o\,(m^{1-k\mu})(1-x)^{-k\mu-1}$$

as $m \to \infty$, uniformly in $0 \leqslant x < 1$, so that (since $k\mu > 0$)

$$\int_0^R Y_m\,dx = o\,(m^{1-k\mu}) \int_0^R (1-x)^{-k\mu-1}\,dx = o\,(m^{1-k\mu})(1-R)^{-k\mu} = o\,(m). \tag{5.10}$$

Again, by partial summation and Lemmas 4 and 3,

$$\sum_{s=m+1}^{\infty} s^{-k\mu}|\tau_s^\alpha|^k x^s \leqslant o\,(1)(1-x) \sum_{s=m+1}^{\infty} E_s^{1-k\mu} x^s$$
$$\leqslant o\,(m^{2-k\mu})(1-x)x^m + o\,(1)(1-x)^{k\mu-1}x^m$$

as $m \to \infty$, uniformly in $0 \leqslant x < 1$, whence

$$Z_m = o\,(m^{2-k\mu})(1-x)^{-k\mu}x^m + o\,(1)(1-x)^{-2}x^m = o\,(m^2)x^m,$$

uniformly in $0 \leqslant x \leqslant R$. Hence also

$$\int_0^R Z_m\,dx = o\,(m^2) \int_0^R x^m\,dx = o\,(m). \tag{5.11}$$

From (5.4)–(5.11), we have now

$$\int_0^R \Big(\sum_0^m (n+1)|\sigma_n^\alpha - \phi(x)|^k x^n \Big)\,dx = o\,(m),$$

and this completes the proof of Theorem 4.

A similar argument proves also Theorem 5, the O-form of Lemma 4 now being used in place of the o-form.

[*Added in proof* 23 *August* 1958.] In the condition (1.9) in Theorem 4 the o can be replaced by O. The proof of this will be given in (4).

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as $m \to \infty$, uniformly in $0 \leqslant x < 1$, so that (since $k\mu > 0$)

$$\int_0^R Y_m\,dx = o\,(m^{1-k\mu}) \int_0^R (1-x)^{-k\mu-1}\,dx = o\,(m^{1-k\mu})(1-R)^{-k\mu} = o\,(m). \tag{5.10}$$

Again, by partial summation and Lemmas 4 and 3,

$$\sum_{s=m+1}^{\infty} s^{-k\mu} |\tau_s^\alpha|^k x^s \leqslant o\,(1)(1-x) \sum_{s=m+1}^{\infty} E_s^{1-k\mu} x^s$$
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